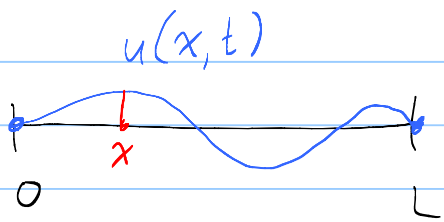
Lecture 8 D'Alembert's solution



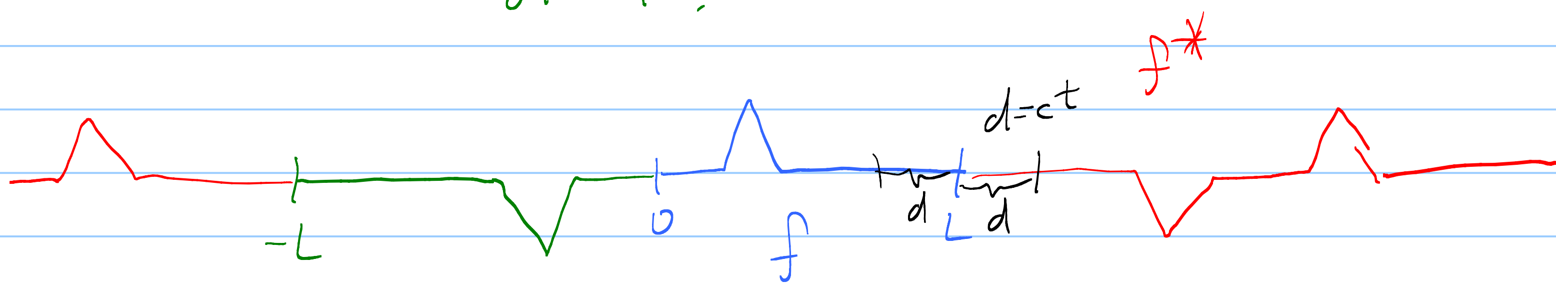
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

BC $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ IC $\quad u(x,0) = f(x)$, $\quad \frac{\partial u}{\partial t}(x,0) = g(x)$

Johann's $\rightsquigarrow$ $u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct) \right]$

[case $g(x) \equiv 0$]

where f^* is the <u>odd</u> periodic extension of f.



Last time: u solves PDE ✓

u solves IC ✓

What about BC?

BC $\begin{cases} u(0,t) = 0 & (1) \\ u(L,t) = 0 & (2) \end{cases}$ $\quad u(0,t) = \frac{1}{2}\underbrace{\left[f^*(0+ct) + f^*(0-ct)\right]}_{\text{Aha! Need}} \overset{\text{want}}{=} 0$

$-f^*(ct) = f^*(-ct)$

f^* needs to be odd to make 1 happen.

$$(2): \frac{1}{2}\left[f^*(L+ct) + f^*(L-ct)\right] \overset{\text{want}}{=} 0$$

Aha! Odd periodic ext makes that happen.

Jacob's nice clean sol^n even when $g \not\equiv 0$.

Step 1: Let f^* and g^* be odd periodic extensions of f and g.

Let $G(x) = \int_0^x g^*(u)\,du$.

Sol^n:
$$\boxed{u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct)\right] + \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]}$$

D'Alemberts: New vars: $v = x+ct$
$w = x - ct$

$$u_{tt} = c^2 u_{xx}$$

$$u_x = \frac{\partial u}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{1} + \frac{\partial u}{\partial w}\underbrace{\frac{\partial w}{\partial x}}_{1} = u_v + u_w$$

$$u_{xx} = \left(\frac{\partial u_v}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u_v}{\partial w}\frac{\partial w}{\partial x}\right) + \left(\frac{\partial u_w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u_w}{\partial w}\frac{\partial w}{\partial x}\right)$$

$$u_{xx} = u_{vv} + \underbrace{u_{wv} + u_{vw}}_{2u_{wv}} + u_{ww} = u_{vv} + 2u_{wv} + u_{ww}$$

$$u_t = \frac{\partial u}{\partial v}\underbrace{\frac{\partial v}{\partial t}}_{c} + \frac{\partial u}{\partial w}\underbrace{\frac{\partial w}{\partial t}}_{-c} = c u_v - c u_w$$

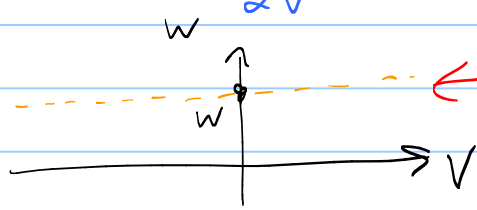
$$\underline{u_{tt} = c^2\left(u_{vv} - 2u_{wv} + u_{ww}\right)} \neq c^2 \cdot$$

Plug into wave eqn: $\quad -2c^2 u_{wv} = 2c^2 u_{wv}$

$$4c^2 u_{wv} = 0$$

$$\boxed{u_{wv} \equiv 0}$$

Sol^n to $\dfrac{\partial u}{\partial v} \equiv 0$ $\quad$ so $\quad u(v,w) \equiv C(w)$



$\leftarrow$ u const on horiz

$C(w)$ is an arbitrary f'cn of w.

Our prob: $\quad \dfrac{\partial}{\partial w}\underbrace{\left[\dfrac{\partial u}{\partial v}\right]} \equiv 0$

$$\boxed{\frac{\partial u}{\partial v} = C(v)}$$

Hmmm. Let $p(v)$ be an antiderivative of $C(v)$.

So $p'(v) = C(v)$.

$$\frac{\partial u}{\partial v} = C(v) = p'(v)$$

$$\frac{\partial u}{\partial v} - p'(v) \equiv 0$$

$$\frac{\partial}{\partial v}\Big[\underbrace{u - p(v)}\Big] \equiv 0$$

So $u - p(v) = K(w)$

Get $u(w,v) = p(v) + K(w)$

and $u(x,t) = \underset{\varphi}{\underset{\uparrow}{p}}(x+ct) + \underset{\psi}{\underset{\uparrow}{K}}(x-ct)$

D'Alembert's: $u(x,t) = \varphi(x+ct) + \psi(x-ct)$

$$\frac{\partial u}{\partial t}(x,t) = \varphi'(x+ct)\cdot c + \psi'(x-ct)(-c)$$

IC: $u(x,0) = \varphi(x) + \psi(x) \overset{\text{want}}{=} f(x)$ (A)

$$\frac{\partial u}{\partial t}(x,0) = c\varphi'(x) - c\psi'(x) = g(x) \quad \text{(B)}$$

Step: Assume $\varphi(0) = 0$, $\psi(0) = 0$.

Integrate B $\int_0^x$

$$c\int_0^x \varphi'(u)\,du - c\int_0^x \psi'(u)\,du = \int_0^x g(u)\,du$$

$$\varphi(x) - \psi(x) = \frac{1}{c} G(x) \qquad (B')$$

$$\boxed{\begin{array}{ll} (A) & \varphi(x) + \psi(x) = f(x) \\ (B') & \varphi(x) - \psi(x) = \frac{1}{c} G(x) \end{array}}$$

$(A) + (B)$: $\quad \varphi(x) = \frac{1}{2}\left[f(x) + \frac{1}{c} G(x) \right]$

$(A) - (B)$: $\quad \psi(x) = \frac{1}{2}\left[f(x) - \frac{1}{c} G(x) \right]$

Last: BC hold!

Jacob's sol^{n} works.