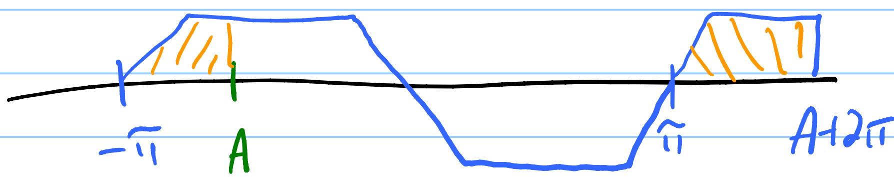
Lecture 9

f C^1-smooth and 2π periodic.

Then F.S. $\to f$ at each x.

Periodic fcn fact: f 2π periodic.

$$\int_{-\pi}^{\pi} f\,dt = \int_A^{A+2\pi} f\,dt \quad \text{for any } A.$$



$$S_N(x) = a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= \sum_{n=-N}^{N} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

$$= \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\underbrace{e^{-int}\cdot e^{inx}}_{e^{inx-int}}\,dt\right)$$

$$= \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\sum_{n=-N}^{N} e^{in(x-t)}\right) f(t)\,dt$$

$D_N(x-t)$

<u>Dirichlet kernel</u> : $D_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx}$

<u>Properties</u> : 1) $D_N(x) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$

2) L'H shows $D_N(0) = \frac{1}{2\pi} \cdot \frac{N+\frac{1}{2}}{\frac{1}{2}}$

3) $\int_{-\pi}^{\pi} D_N(t)\,dt = \underline{1}$

4) $D_N(x)$ is 2π periodic, even

Bad thing : $\int_{-\pi}^{\pi} |D_N(t)|\,dt \to \infty$ as $N \to \infty$.

Pf of 3: $\displaystyle\int_{-\pi}^{\pi} \frac{1}{2\pi}\sum_{-N}^{N} e^{int}\,dt = \sum \int$

$$= 0 + \cdots + 0 + \int_{-\pi}^{\pi} \frac{1}{2\pi}\cdot 1\,dt + 0 + \cdots + 0 = 1$$

Geometric series: $1 + z + z^2 + \cdots + z^N = \sigma_N$

$$\begin{array}{r} z + z^2 + \cdots + z^N + z^{N+1} = z\sigma_N \\ \hline 1 - z^{N+1} \qquad = (1-z)\sigma_N \end{array}$$

$$1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}$$

Aha! $z = e^{ix}$, $z^2 = e^{i2x}$, $z^3 = e^{i3x}$, ...

$$\boxed{1 + e^{ix} + e^{i2x} + \cdots + e^{iNx} = \frac{1 - e^{i(N+1)x}}{1 - e^{ix}}}$$

The rest: $e^{-ix} + e^{-2ix} + \cdots + e^{-Nix}$

$$= \underset{\uparrow\ \frac{1}{e^{ix}}}{e^{-ix}} \left[\underbrace{1 + e^{-ix} + \left(e^{-ix}\right)^2 + \cdots + \left(e^{-ix}\right)^{N-1}}_{\left(\frac{1 - e^{-iNx}}{1 - e^{-ix}}\right)} \right]$$

$$\cdot \left(\frac{1 - e^{-iNx}}{1 - e^{-ix}}\right)$$

$$= \frac{1 - e^{-iNx}}{e^{ix} - 1}$$

$$D_N(x) = \frac{1 - e^{i(N+1)x}}{1 - e^{ix}} + \frac{1 - e^{-iNx}}{e^{ix} - 1}$$

$$= \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}} \cdot \frac{e^{-ix/2}}{e^{-ix/2}}$$

$$= \frac{(-1)}{(-1)} \cdot \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-ix/2} - e^{ix/2}} \cdot \frac{2i}{2i}$$

$$= \frac{\sin (N+\frac{1}{2})x}{\sin \frac{x}{2}}$$

$$S_N(x) - f(x) =$$

$$\underbrace{\int_{-\pi}^{\pi} D_N(\underbrace{x-t}_{\tau}) f(t)\, dt}_{I} - f(x) \underbrace{\int_{-\pi}^{\pi} D_N(t)\, dt}_{1}$$

$$\tau = x - t \quad \leftarrow t = x - \tau$$

$$d\tau = -dt$$

$$I - \int_{\pi}^{\pi} D_N(t) f(x)\, dt$$

when $t = -\pi$: $\tilde{\tau} = x + \pi$

$t = \pi$: $\tilde{\tau} = x - \pi$

$$I = \int_{\tilde{\tau} = x+\pi}^{x-\pi} D_N(\tilde{\tau}) f(x - \tilde{\tau}) \, (-d\tilde{\tau})$$

$$= \int_{x-\pi}^{x+\pi} \underbrace{D_N(\tilde{\tau}) f(x - \tilde{\tau})}_{2\pi \text{ periodic}} \, d\tilde{\tau}$$

$$= \int_{-\pi}^{\pi} D_N(t) f(x - t) \, dt$$

Now $S_N(x) - f(x) =$

$$= \int_{-\pi}^{\pi} D_N(t) \left[f(x-t) - f(x) \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\left(N + \tfrac{1}{2}\right)t \; \underbrace{\frac{f(x-t) - f(x)}{\sin \frac{t}{2}}}_{g(t)} \, dt$$

L'H shows g is cont.

Hmmm. $\sin\left(N+\frac{1}{2}\right)t = \sin Nt \cos\frac{t}{2} + \cos Nt \sin\frac{t}{2}$

$$S_N(x)-f(x) = \underbrace{\int_{-\pi}^{\pi}(\sin Nt)\left[\underline{\frac{1}{2\pi}\cos\frac{t}{2}\, g(t)}\right]dt}_{b_N \text{ of F.S. for}}$$

$$+ \underbrace{\int_{-\pi}^{\pi}\cos Nt\left[\underline{\frac{1}{2\pi}\sin\frac{t}{2}\, g(t)}\right]dt}_{a_N \text{ of F.S. for}}$$

Last step: Bessel's Ineq $\Longrightarrow$

Fourier coeff $\longrightarrow 0$ as $N \to \infty$.

Remark: Can be localized. All I needed was continuity of f and diff'billity at x. In fact, all I needed was an estimate:

$$|f(x-t) - f(x)| \leq C|t|$$

Lipshitz estimate a x. (MVT $\Rightarrow$ Lip 1)