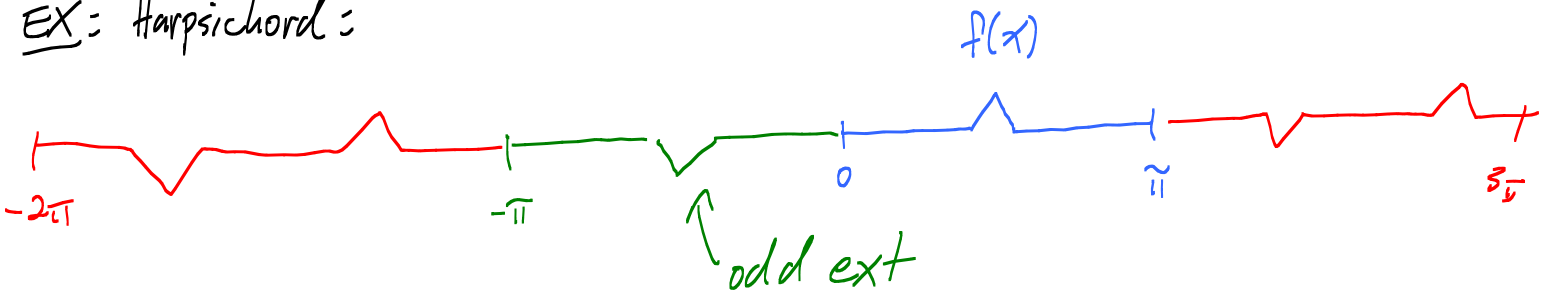
# Lecture 4    Fourier sine series    $\sum_{n=1}^{\infty} A_n \sin nx$

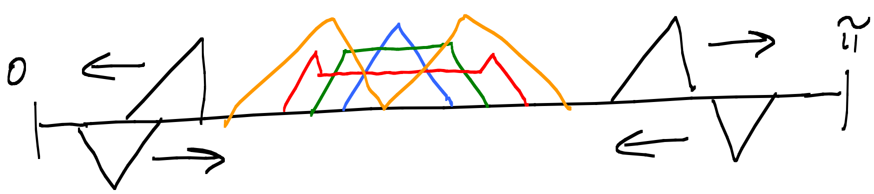
Homework 1 problems on Home page due next Monday.

EX: Harpsichord:

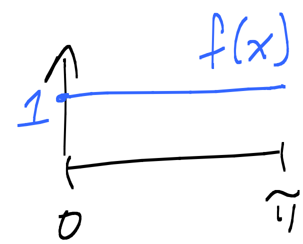



$2\pi$-periodic extension.

Jacob's sol$^n$: $\frac{1}{2}\left[f(x+t)+f(x-t)\right]$




Prob: Find the Fourier sine series for




$$f(x) \overset{?}{=} \sum_{n=1}^{\infty} A_n \sin nx$$

where $$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

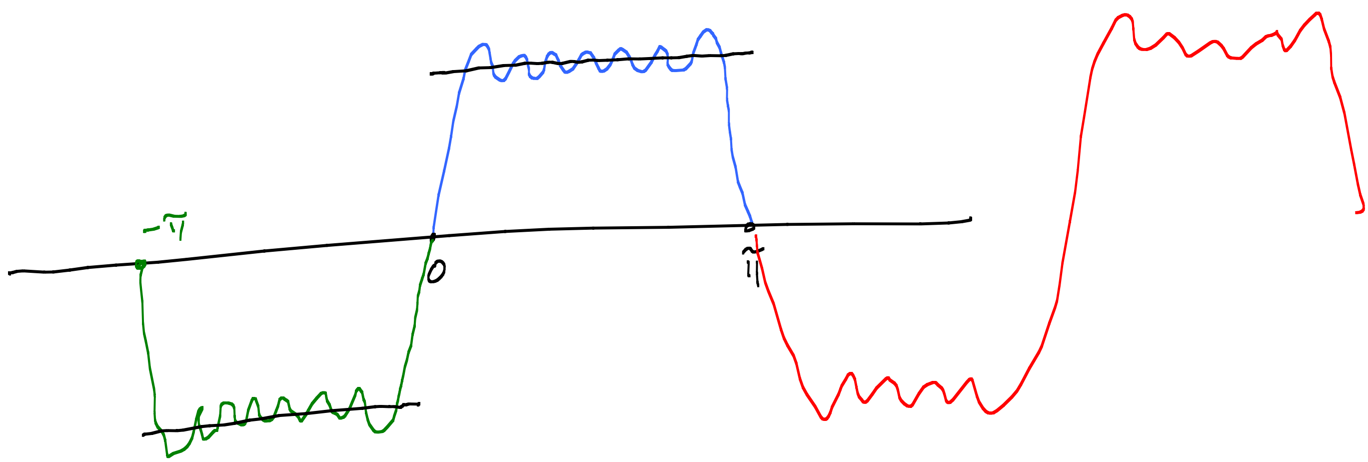
$$= \frac{2}{\pi n}\left[1 - \cos n\pi\right]$$

$$= \frac{2}{\pi n}\left[1-(-1)^n\right] = \begin{cases} \frac{4}{n\pi} & \text{when } n=\text{odd} \\ 0 & n=\text{even} \end{cases}$$

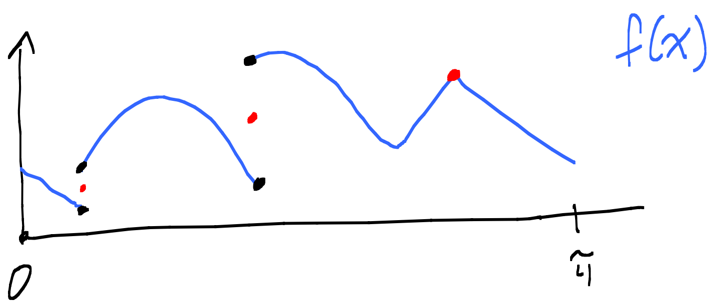
$$\frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

Ouch! $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots = \infty.$

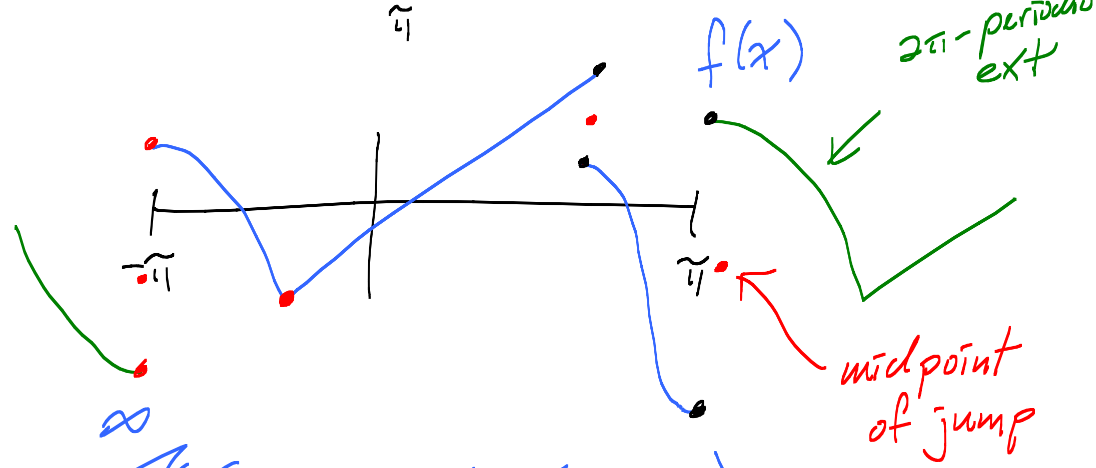
But F.S.S. does converge — slowly!




Big Fact: Fourier Sine series converges to $f(x)$ nicely and fast where $f$ is continuously diff'ble in $(0,\pi)$, and converges, but slowly to $f(x)$ where $f$ has a corner, and converges to the midpoint of jumps. It converges to zero at $0$ and $\pi$.




## Full Fourier Series:




$$f(x) \stackrel{?}{=} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$

Key: $\{1, \sin nx, \cos nx\}$ are orthogonal

$$\int_{-\pi}^{\pi} \underbrace{\sin}_{\cos} nx \underbrace{\sin}_{\cos} mx \, dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

$$\int_{-\pi}^{\pi} 1 \cdot \underbrace{\sin nx}_{\cos nx} \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad \text{for all } n, m$$

Hmmm $$\int_{-\pi}^{\pi} f(x) \cos 3x \, dx \stackrel{?}{=} \int_{-\pi}^{\pi} \left( a_0 \cos 3x + \sum a_n \cos nx \cos 3x + \cdots \right) dx$$

$$= a_3 \int_{-\pi}^{\pi} \cos^2 3x \, dx$$

$$= \pi a_3$$

$$\int_{-\pi}^{\pi} f(x)\cdot 1 \, dx = \int_{-\pi}^{\pi} a_0 \, dx + \sum \left( a_n \underbrace{\int_{-\pi}^{\pi} \cos nx \, dx}_{=0} + \cdots \right)$$

$$= a_0 \cdot 2\pi$$

**Famous formulas:**

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad \leftarrow \text{ave!}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad n = 1, 2, 3 \ldots$$

**Facts:** If $f$ is odd, all $a_n$ terms vanish. ← sine series

If $f$ is even, all $b_n$ terms vanish. ← Cosine series

**Why:**

$$\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}} \cdot \underbrace{\cos nx}_{\text{even}}}_{\text{odd}} \, dx = 0$$

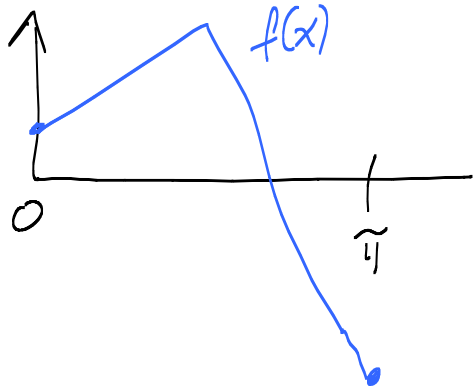
$$\int_{-L}^{L} \text{Odd} \, dx = 0 \qquad \int_{-L}^{L} \text{Even} \, dx = 2 \int_{0}^{L} \text{Even} \, dx$$

odd · even = odd

odd · odd = even

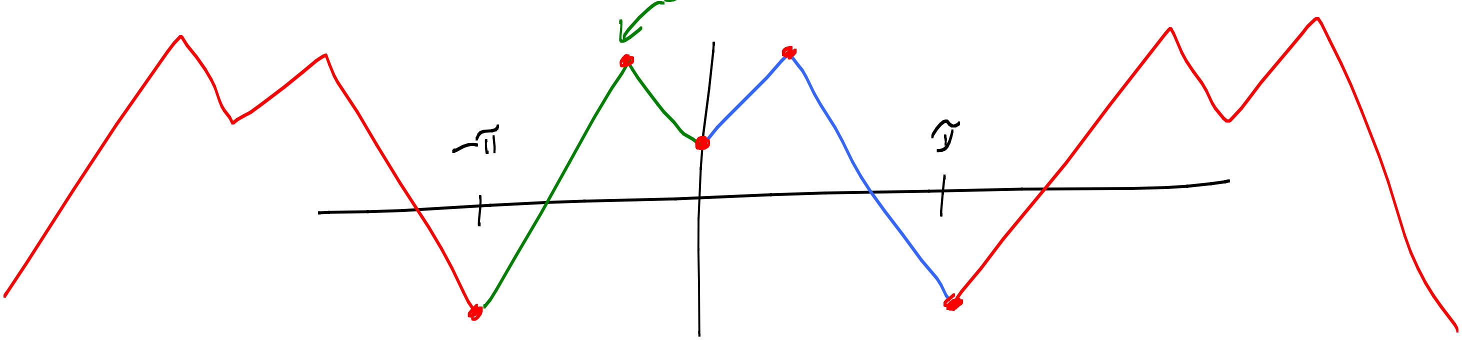
even · even = even

## Fourier Cosine Series :




$$f(x) = \sum_{n=0}^{\infty} A_n \cos nx \qquad A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$




Civilized way :

$$f(x) \overset{?}{=} \sum_{n=-\infty}^{\infty} c_n e^{inx}$$