Lecture 5 Complex notation for Fourier series

Complex exponential $E(x+iy) = e^x \cos y + i e^x \sin y$

Facts about $E(z) = e^z$: 0) $E(0) = 1$

1) $E(z)$ is complex diff'ble $\lim_{z \to a} \frac{E(z) - E(a)}{z-a}$ exists

[call it $E'(z)$]. And $E'(z) = E(z)$!

2) $E(z+w) = E(z)E(w)$ ← Pf: Just use $e^{x_1+x_2} = e^{x_1}e^{x_2}$ and sum of angles trig formulas.

3) $E(x) = e^x$ when $x \in \mathbb{R}$. ← $E(z)$ is the only $\mathbb{C}$-diff'ble fcn on $\mathbb{C}$ agreeing with e^x on $\mathbb{R}$.

4) $E(i\theta) = \cos\theta + i \sin\theta$ and (2) yields de'Moivre's

$$\left(e^{i\theta}\right)^N = e^{iN\theta}$$

5) $$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

(+): $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

(−): $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

Me: $y'' + k^2 y = 0$

Try $y = e^{rx}$. $y' = re^{rx}$, $y'' = r^2 e^{rx}$

Plug in $r^2 e^{rx} + k^2 e^{rx} \overset{\text{want}}{=} 0$

$$(r^2+k^2)\underbrace{e^{rx}}_{\text{never zero}} \overset{\text{want}}{=} 0$$

Need $\boxed{r^2+k^2=0}$

So $r=\pm ik$.

Me: e^{ikx} is a complex sol^n

$= \cos kx + i \sin kx$

Verify: $\underbrace{(\cos kx + i\sin kx)''}_{-k^2\cos kx + i(-k^2\sin kx)} + k^2(\cos kx + \sin kx) = 0$

Aha! $[u+iv]'' + k^2[u+iv] = 0$

$$\underbrace{[u''+k^2u]}_{} + i\underbrace{[v''+k^2v]}_{} = 0$$

$u = \mathrm{Re}\, e^{ikx} = \cos kx$ $\qquad v = \mathrm{Im}\, e^{ikx} = \sin kx$

are two real sol^{n}s.

$y = c_1 \cos kx + c_2 \sin kx$ is a real sol^n.

Can solve any I.V.P. So it is the gen^l sol^n.

Remark: Hmmm. e^{-ikx} is also a complex sol^n.

Get $y = c_1\cos kx + c_2(-\sin kx)$ ← same real sol^n

Seeley: Get two complex sol^{n}s $y = e^{ikx}, e^{-ikx}$

Get more complex sol^{n}s $\underbrace{y = c_1 e^{ikx} + c_2 e^{-ikx}}_{y}$

EX: Solve $y'' + k^2 y = 0$

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$$y' = c_1 ik e^{ikx} - c_2 ik e^{-ikx}$$

want

$$c_2 = -c_1 \rightarrow \begin{cases} y(0) = c_1 + c_2 = 0 \\ y'(0) = c_1 ik - c_2 ik = 1 \end{cases}$$

$$c_1 ik - (-c_1) ik = 1$$

$$c_1 = \frac{1}{2ik} \quad c_2 = -\frac{1}{2ik}$$

So $y = \frac{1}{2ik} e^{ikx} - \frac{1}{2ik} e^{-ikx}$

$$= \frac{1}{k}\left(\frac{e^{ikx} - e^{-ikx}}{2i}\right) = \frac{1}{k} \sin kx \quad \checkmark$$

Complex Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \underbrace{\cos nx}_{\frac{e^{inx}+e^{-inx}}{2}} + b_n \underbrace{\sin nx}_{\frac{e^{inx}-e^{-inx}}{2i}}\right) \quad \text{on } [-\pi, \pi]$$

Need

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\left(a_n + \frac{b_n}{i}\right) \right) e^{inx} + \left(\frac{1}{2}\left(a_n - \frac{b_n}{i}\right)\right) e^{-inx} \right]$$

Hmmm. $$\underbrace{\frac{1}{2}\left(a_n + \frac{b_n}{i}\right)}_{c_n} = \frac{1}{2} \cdot \frac{1}{\pi} \left[\int_{-\pi}^{\pi} \underbrace{f(x) \cos nx \, dx - i\, f(x) \sin nx}_{f(x) e^{-inx}} \, dx \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx} \, dx$$

Same with $(-n)$ in place of n!

Wow! $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

Another way to think about it: Complex inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx$$

Fact: $\{e^{inx}\}_{i=-\infty}^{\infty}$ are orthogonal.

$$\int_{-\pi}^{\pi} e^{inx} \overline{e^{imx}} \, dx = \int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx$$

$$= \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx$$

$$= \underbrace{\int_{-\pi}^{\pi} \cos(n-m)x \, dx}_{=0 \text{ if } n \neq m} + i \underbrace{\int_{-\pi}^{\pi} \sin(n-m)x \, dx}_{=0 \text{ if } n \neq m.}$$

$$n = m: \quad \langle e^{inx}, e^{inx} \rangle = \int_{-\pi}^{\pi} \underbrace{e^{i\cdot 0x}}_{1} \, dx = 2\pi$$

Def^n: Trigonometric polynomials:

$$a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) = \sum_{n=-N}^{N} c_n e^{inx}$$

Trig poly of degree N

Why poly? $\left(e^{inx}\right) = \left(e^{ix}\right)^n$

Bessel's inequality:

In the context of Fourier series: The partial sum

$$\sum_{-N}^{N} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \underbrace{e^{-inx}}_{\overline{e^{inx}}} \, dx$$

is the best approximation to f among trig polys of degree N in $L^2[-\pi, \pi]$.

Def^n: $\langle f, g \rangle$ = complex inner product. (L^2-inner prod)

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\pi}^{\pi} |f(x)|^2 \, dx}$$

Fact: $\left\| f - \sum_{-N}^{N} c_n e^{inx} \right\|^2 \leq \left\| f - \sum_{-N}^{N} A_n e^{inx} \right\|^2$

Fourier coeff $\qquad$ any coeff

and only get equality when $c_n = A_n$, $n = 0, \pm 1, \pm 2, \ldots, \pm N$.

Bessel's Ineq: $f(x) = \sum_{n=1}^{\infty} c_n \varphi_n$ where φ_n orthonormal

meaning $\int_{-\pi}^{\pi} \varphi_n \overline{\varphi_m} \, dx = 0$ if $n \neq m$ and

$$\int_{-\pi}^{\pi} |\varphi_n|^2 \, dx = \underline{\underline{1}}$$

normalized.

Fourier series: $\varphi_n = \frac{1}{\sqrt{2\pi}} e^{inx}$