Lecture 7: Dirichlet problem for the unit disc

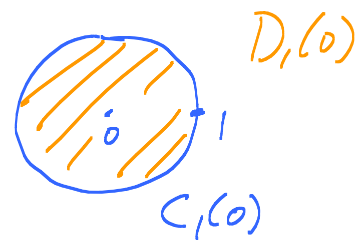
HWK 2 due Wed.

Steady state heat equation: Unit disc $D_1(0)$ is a metal plate.

We know temp on edges: unit circle $C_1(0)$

u = temp. Know u on $C_1(0)$.

Want u on $D_1(0)$ after "steady state".

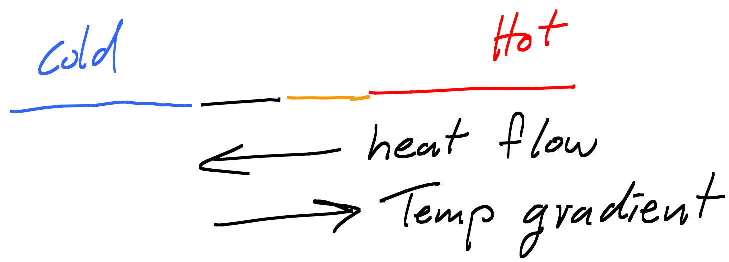


Heat equation: $$\underbrace{\frac{\partial u}{\partial t}}_{\substack{\downarrow \\ 0}} = c\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{\Delta u}$$

as $t \to \infty$ for "steady state"

Steady state heat equ: $\Delta u = 0$, Laplace equ.

Derive heat eqn: $\mathbb{R}^3$.



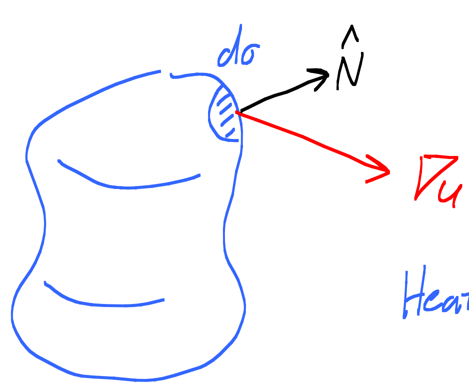
Kelvin's Law: (Heat flow) $= -c$ (temp gradient)

"Control volume"

blob

Volume V

boundary surface S



Heat flow in $=$

$c\,\nabla u \cdot \hat{N}\, d\sigma$

Net heat flow in : $c \iint\limits_S \nabla u \cdot \hat{N} \underbrace{d\sigma}_{\text{"d Surface area"}}$

Note: One calorie is the amount of heat needed to raise one 1 cm^3 of water one degree.

Total heat in control volume $= k \iiint\limits_V u \, dV$

Aha! $$\frac{\partial}{\partial t}\left(k \iiint\limits_V u \, dV\right) = \underbrace{c \iint\limits_S \nabla u \cdot d\vec{\sigma}}_{= c \iiint\limits_V \underbrace{\mathrm{div}\, \nabla u}_{\Delta u} \, dV}$$

Divergence Thm : $$\iint\limits_S \vec{F} \cdot d\vec{\sigma} = \iiint\limits_V \mathrm{div}\, \vec{F} \, dV$$

$$\mathrm{div}\left(F_1 \hat{\imath} + F_2 \hat{\jmath} + F_3 \hat{k}\right) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

$$\mathrm{div}\left(\frac{\partial u}{\partial x_1}\hat{\imath} + \frac{\partial u}{\partial x_2}\hat{\jmath} + \frac{\partial u}{\partial x_3}\hat{k}\right) = \Delta u$$

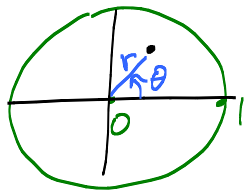
Now $$\underbrace{\frac{\partial}{\partial t}\left(k \iiint\limits_V u \, dV\right)}_{k \iiint\limits_V \frac{\partial u}{\partial t} \, dV} = c \iiint\limits_V \Delta u \, dV$$

Aha! $$\iiint_V \Big(\underbrace{\kappa \frac{\partial u}{\partial t} - c\,\Delta u}\Big)\, dV = 0$$

True for any V! This must be zero.

Get heat equ.

Dirichlet prob:



$u(r,\theta)$ = temp.

Given boundary temp $f(\theta)$.

Want u continuous on $\{(x,y): x^2+y^2 \le 1\}$ such that

$$\Delta u \equiv 0 \text{ on } \{(x,y): x^2+y^2 < 1\},$$ ← harmonic

and $$u(1,\theta) = f(\theta).$$

Polar form of $$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Big idea Try $u(r,\theta) = R(r)\,\Theta(\theta)$.

Want $$\frac{\partial^2}{\partial r^2}(R\Theta) + \frac{1}{r}\frac{\partial}{\partial r}(R\Theta) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(R\Theta) \overset{\text{want}}{=} 0$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Mult by r^2: $$r^2R''\Theta + rR'\Theta = -R\Theta''$$

Divide by $R\Theta$: $$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

[Take $\theta = 0$. LHS = const. RHS = same const.

Both must be a constant, λ.

Get two ODE's : $\underbrace{\Theta'' + \lambda \Theta = 0}$, $r^2R'' + rR' - \lambda R = 0$

Hidden boundary conditions!

$\theta \in [-\pi, \pi]$. Need $\begin{cases} \Theta(-\pi) = \Theta(\pi) & \leftarrow \text{no jumps} \\ \Theta'(-\pi) = \Theta'(\pi) & \leftarrow \text{no kinks} \end{cases}$

"Periodic boundary conditions."