Lecture 9 The Poisson integral formula for the sol^n to the Dirichlet Prob on the unit disc.

ODEs : 1) $ay'' + by' + cy = 0 \leftarrow$ 2nd order linear homog ODE with const coeff.

Try $y = e^{rx}$.

2) Euler equ: $ax^2y'' + bxy' + cy = 0$

Try $y = x^p$. Plug in: $ax^2[p(p-1)x^{p-2}] + bx[px^{p-1}] + cx^p \overset{\text{want}}{=} 0$

$$\Big[\underbrace{ap(p-1) + bp + c}_{\text{need this } = 0.}\Big]x^p = 0$$

Cases: Real roots p_1, p_2 with $p_1 \neq p_2$.

$$y = c_1 x^{p_1} + c_2 x^{p_2} \leftarrow \text{gen}^l \text{ sol}^n$$

Case: Repeated root p (real).

Hmmm. Get only sol^n $y = cx^p$. $\leftarrow y_1 = x^p$

Method of "Reduction of order." Try $y_2 = u\, y_1$.

Plug in and use fact that y_1 solves ODE. Stuff cancels. Get first order ODE for u'.

For Euler equ, find that $u = \ln|x|$ works.

Gen^l Sol^n is $y = c_1 x^p + c_2 x^p \ln|x|$

Case: $p = a \pm bi$

Get complex sol^n using $p = a+bi$:

need $x>0$

$$x^{a+bi} = x^a x^{bi} = x^a \left(e^{\ln x}\right)^{bi} = x^a e^{ib\ln x}$$

$$= \underline{x^a \cos(b\ln x)} + i\, \underline{x^a \sin(b\ln x)}$$

Get 2 real sol^{n}s from 1 complex sol^n!

Gen^l Sol^n: $y = c_1 x^a \cos(b\ln x) + c_2 x^a \sin(b\ln x)$ ← $x>0$

Dirichlet prob on disc: Given continuous function $f(\theta)$ on $[-\pi,\pi]$ such that $f(-\pi) = f(\pi)$, find a $u(r,\theta)$ continuous on $\{(r,\theta) : r \le 1\}$ and harmonic on $\{(r,\theta) : r < 1\}$ and having boundary values given by f: $u(1,\theta) = f(\theta)$.

Last time: Found $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$

where, to make $u(1,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) = f(\theta)$

we must have
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta$$

Next step: Play around and get a nicer formula.

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$ ← equiv complex way to write Fourier series.

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,dt\right)}_{c_n} e^{in\theta}$$

$$= \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}\right)}_{P(re^{i\theta},\, e^{it})} f(t)\,dt$$

$P(\underbrace{re^{i\theta}}_{z}, e^{it})$ ← Poisson kernel

z in unit disc; e^{it} ← point on unit circle

$$P_N(re^{i\theta}, e^{it}) = \frac{1}{2\pi} + \underbrace{\frac{1}{2\pi}\sum_{n=1}^{N} r^n e^{in(\theta-t)}}_{n \text{ positive terms}} + \underbrace{\frac{1}{2\pi}\sum_{n=1}^{N} r^n e^{-in(\theta-t)}}_{\text{rewritten } n \text{ neg terms.}}$$

$$= \frac{1}{2\pi}\left[1 + \sum_{n=1}^{N}\left[\underbrace{re^{i(\theta-t)}}_{z}\right]^n + \sum_{n=1}^{N}\left[re^{-i(\theta-t)}\right]^n\right]$$

Aha! Geometric series! $1+z+z^2+\cdots+z^N = \frac{1}{1-z} - \frac{z^{N+1}}{1-z}$

$$\sum_{n=1}^{N} z^n = z \sum_{n=0}^{N-1} z^n = \underbrace{\frac{z}{1-z}}_{\text{limit}} - \underbrace{\frac{z^{N+1}}{1-z}}_{\substack{\text{Error} \\ \varepsilon}}$$

$$|\varepsilon| \le \frac{|z^{N+1}|}{|1-z|} \le \frac{|z|^{N+1}}{1-|z|} \le \frac{r^{N+1}}{1-r} \longrightarrow 0 \quad \text{if } r<1.$$

Get

$$P(re^{i\theta}, e^{it}) = \frac{1}{2\pi}\left[1 + \frac{re^{i(\theta-t)}}{1-re^{i(\theta-t)}} + \frac{re^{-i(\theta-t)}}{1-re^{-i(\theta-t)}}\right]$$

$$= \frac{1}{2\pi} \frac{1-r^2}{|1-re^{i(\theta-t)}|^2} \cdot \underbrace{\frac{1}{|e^{it}|^2}}_{1}$$

Poisson kernel: $\boxed{P(re^{i\theta}, e^{it}) = \frac{1}{2\pi} \frac{1-r^2}{|re^{i\theta}-e^{it}|^2}}$

Sol^n to D. Prob $\quad u(r,\theta) = \int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) f(t)\, dt$

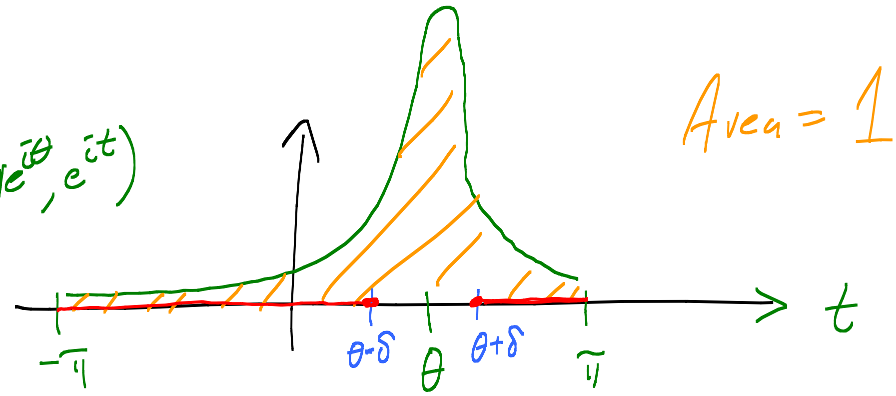
Poisson integral formula

Poisson kernel is a "good" kernel.

1) $P > 0$

2) $\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\, dt = 1$ when $r<1$.

3) $P(re^{i\theta}, e^{it})$



The closer r is one, the more the area is concentrated at θ.

$P(re^{i\theta}, e^{it}) \to 0$ uniformly on

$I_\delta = [-\pi, \pi] - (\theta-\delta, \theta+\delta)$ as $r \to 1$.

and 4) $P(re^{i\theta}, e^{it})$ is harmonic in (r, θ).