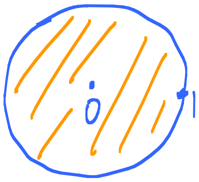
Lecture 10 Poisson integral formula for the sol^n to the Dirichlet Prob.



$u(r,\theta)$ harmonic on $D_1(0)$ such that

$u(1,\theta) = f(\theta)$ is given.

Got $$u(r,\theta) = \underset{\substack{\uparrow \\ \Delta 1 = 0}}{a_0} + \sum_{n=1}^{\infty} \Big(a_n \underbrace{r^n \cos n\theta}_{\Delta = 0} + b_n \underbrace{r^n \sin n\theta}_{\Delta = 0} \Big)$$

Plug in $r=1$. See that coeff should be just the Fourier coeff for $f(\theta)$.

Algebra, simplification via $e^{in\theta}$, limits :

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \frac{e^{in\theta} + e^{-in\theta}}{2} + b_n \frac{e^{in\theta} - e^{-in\theta}}{2i} \right)$$

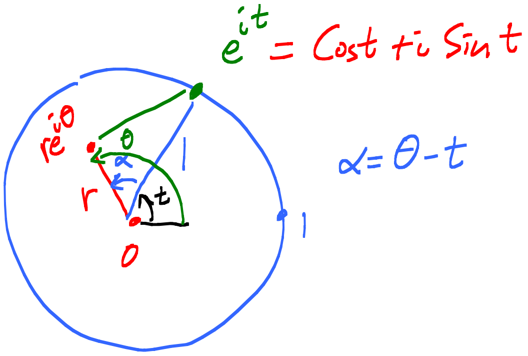
$$= \underset{\substack{\uparrow \\ c_0}}{a_0} + \sum_{n=1}^{\infty} r^n \underbrace{\left(\frac{a_n - b_n i}{2} \right)}_{c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta} e^{in\theta} + \sum_{n=1}^{\infty} r^n \underbrace{\left(\frac{a_n + b_n i}{2} \right)}_{c_{-n} = \overline{c_n}} e^{-in\theta}$$

$\uparrow$ f real

$$= \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$ $\longleftarrow$ use Geometric series formula.

Poisson integral formula :

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\frac{1 - r^2}{|re^{i\theta} - e^{it}|^2}}_{P(re^{i\theta}, e^{it})} f(t)\, dt$$



$\alpha = \theta - t$

Law of cosines

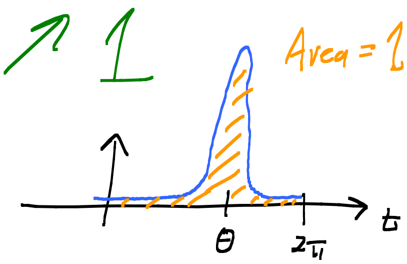
$$(\text{green line})^2 = 1^2 + r^2 - 2r \,\text{Cos}(\theta - t)$$

4 big properties: 1) $P(re^{i\theta}, e^{it}) > 0$ ✓ formula

2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\, dt = 1$ when $r < 1$.

3) $P(re^{i\theta}, e^{it}) \to 0$ unformly as $r \nearrow 1$

for $t \in [-\pi, \pi] - (\theta - \delta, \theta + \delta)$



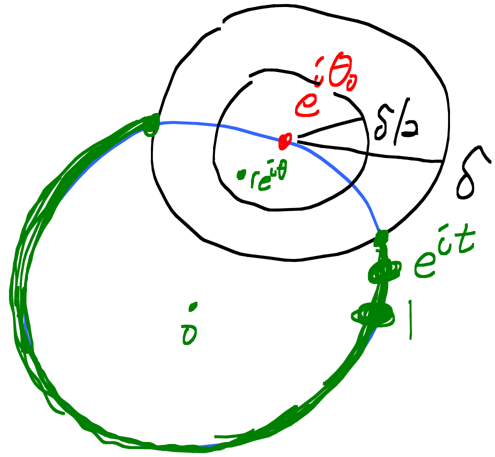
4) $\Delta P = 0$ in (r, θ) when t is held fixed.

Pf: 1. ✓ from formula.

4. ✓ from formula

2. $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{P(re^{i\theta}, e^{it})}_{\sum_{n=-\infty}^{\infty} r^{|n|} \underbrace{e^{in(\theta - t)}}_{\text{integrate to zero if } n \neq 0.}}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} r^0 \cdot 1 \, dt = 1$$

3)



$I_\delta = \{t : e^{it} \notin D_\delta(e^{i\theta_0})\}$

Keep $re^{i\theta}$ in $D_{\delta/2}(e^{i\theta_0})$

Claim: $P(re^{i\theta}, e^{it}) \to 0$ unif for $t \in I_\delta$ as $r \to 1$

$$P(re^{i\theta}, e^{it}) = \frac{1-r^2}{\underbrace{|re^{i\theta} - e^{it}|^2}_{|\;| > \frac{\delta}{2}}} \le \frac{\overbrace{1-r^2}^{(1-r)(1+r)}}{\left(\frac{\delta}{2}\right)^2} < \frac{2(1-r)}{\left(\frac{\delta}{2}\right)^2}$$

$$\frac{1}{2\pi} P(re^{i\theta}, e^{it}) \le \frac{8}{\pi\delta^2}(1-r) \to 0 \quad \text{as } r \nearrow 1.$$

Schwarz Theorem: Poisson formula solves the D. prob.

Pf: $\Delta u \equiv 0$ because we can differentiate under $\int_{-\pi}^{\pi}$ in r, θ variables, and $\Delta P \equiv 0$.

$$u(r,\theta) - f(\theta_0) =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it}) f(t)\, dt - \underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\, dt\right)}_{=1 \text{ (Prop 2)}} f(\theta_0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\left[f(t) - f(\theta_0)\right] dt$$

$$= \underbrace{\frac{1}{2\pi}\int_{I_\delta} \underbrace{P(re^{i\theta}, e^{it})}_{\text{small}} [f(t)-f(\theta_0)]\,dt}_{I_1} + \underbrace{\frac{1}{2\pi}\int_{\theta-\delta}^{\theta+\delta} P(re^{i\theta}, e^{it}) \underbrace{[f(t)-f(\theta_0)]}_{\text{small}}\,dt}_{I_2}$$

Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(t) - f(\theta)| < \varepsilon$ if $|t-\theta| < \delta$.

And, f continuous on closed and bounded $[-\pi, \pi] \Rightarrow$

f assumes max and min there. So $\exists$ bound $M > 0$ such that $|f| < M$ on $[-\pi, \pi]$.

Note: $|f(t) - f(\theta)| \le |f(t)| + |f(\theta)| \le 2M$

$$|I_1| = \frac{1}{2\pi}\left|\int_{I_\delta} P(re^{i\theta}, e^{it})[f(t)-f(\theta_0)]\,dt\right| \le \frac{1}{2\pi}\int |\ \cdots|$$

$$\le \frac{1}{2\pi} \cdot \frac{8(1-r)}{\delta^2} \cdot 2M \cdot \underbrace{\text{length}(I_\delta)}_{< 2\pi}$$

$\longrightarrow 0$ as $r \nearrow 1$.

$$|I_2| = \frac{1}{2\pi}\left|\int_{\theta_0-\delta}^{\theta_0+\delta} P(re^{i\theta}, e^{it})[f(t) - f(\theta_0)]\,dt\right|$$

$$\le \frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} \underbrace{P(re^{i\theta}, e^{it})}_{>0} \underbrace{|f(t) - f(\theta_0)|}_{\varepsilon}\,dt$$

$$\leq \varepsilon \cdot \frac{1}{2\pi} \underbrace{\int_{\theta_0-\delta}^{\theta_0+\delta} P(re^{i\theta}, e^{it})\,dt}_{< \int_0^{2\pi} P(re^{i\theta}, e^{it})\,dt \underset{\#3}{=} 2\pi} < \varepsilon.$$

Conclude $u(r,\theta_0) \to f(\theta_0)$ as $r \nearrow 1$, with $re^{i\theta}$ staying in $D_{\delta/2}(e^{i\theta_0})$.