Lecture 11 Harmonic functions – the averaging property

HWK 3 due Wed.

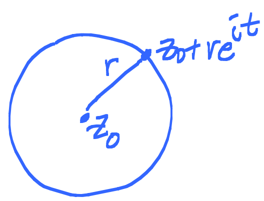
Poisson formula at $r=0$:

$$u(\text{Origin}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-0^2}{\underbrace{|0-e^{it}|^2}_{1}} f(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt \leftarrow \text{Average temp on } C_1(0)!$$

Fact: Harmonic fcns satisfy the Averaging property:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})\,dt$$



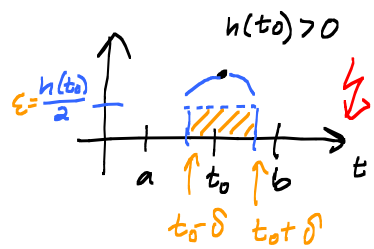
$z_0 \in \mathbb{C} \approx \mathbb{R}^2$

Why: Green's Identities used to show this.

Fact: Ave property $\Longrightarrow$ maximum principle.

Calculus lemma: 1) Suppose $h(t) \geq 0$ is continuous on $[a,b]$.

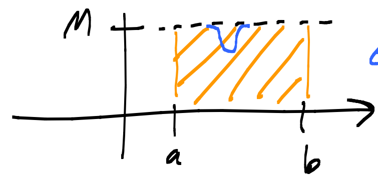
Then $\int_a^b h(t)\,dt = 0 \Longrightarrow h \equiv 0.$



2) Suppose h continuous and $h(t) \leq M$ on $[a,b]$.

If $\int_a^b h\,dt = M(b-a)$, then $h(t) \equiv M$ on $[a,b]$.

Pf: Apply (1) to $M - h(t)$.



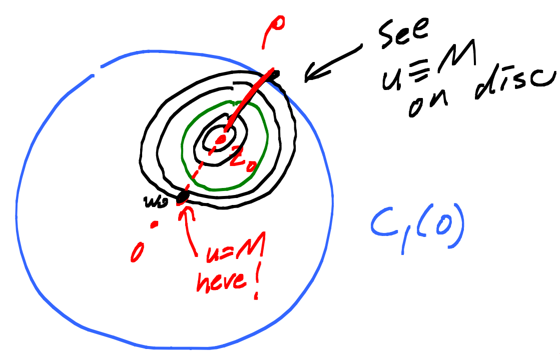
Max Princ for harmonic fcns: 1) u harmonic on $D_1(0)$ and continuous on $\overline{D_1(0)}$. If u assumes its max value M at a point z_0 inside $D_1(0)$, then $u \equiv M$ on $\overline{D_1(0)}$.

2) Cor: Max value of u on $\overline{D_1(0)}$ is attained on boundary.

$$u(z) \le \max_{C_1(0)} u$$

z inside $D_1(0)$

Pf: Suppose $u(z_0) = M$, the max.



continuous $\longrightarrow$ $u(z_0 + re^{it}) \le M$

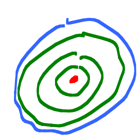
Must $\equiv M$!

$$\underset{M}{\overset{\shortparallel}{u(z_0)}} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + re^{it})}_{\le M}\,dt$$

See $u(z_0 + re^{it}) \equiv M$.

Now let r vary. Aha! Can repeat at w_0. Get radii $\rho, 2\rho, 4\rho, 8\rho, \ldots$

Repeat until $0 \in$ disc where $u \equiv M$. Done! Can use circles centered at origin and fill out $D_1(0)$



$u \equiv M$ on $D_1(0)$.

Cor: Sol^n to D. Prob is unique.

Pf: Suppose u_1 and u_2 solve D. Prob for bndry values f.

Then u_1-u_2 is haronic in $D_1(0)$, cont on $\overline{D_1(0)}$,

and $u_1-u_2 \equiv 0$ on $C_1(0)$.

Max Princ $\Rightarrow$ $u_1-u_2 \leq 0$ on $\overline{D_1(0)}$.

Aha! Can same thing for u_2-u_1. $u_2-u_1 \leq 0$, too.

So $u_1 \equiv u_2$. ✓

Cor: Harmonic fcns are C^∞-smooth.

Pf: Uniqueness $\Rightarrow$ Harmonic fcns $=$ $\underbrace{\text{Poisson integrals}}_{\text{can differentiate under } \int \text{ at will.}}$

Cor: Continuous fcns that satisfy the averaging property are harmonic!

Pf: Suppose u_1 satisfies Ave prop. Ave prop $\Rightarrow$ max princ.

Let u_2 = Poisson formula with $f(\theta) = u_1(e^{i\theta})$.

u_2 also satisfies Ave prop. $u_1 - u_2 \leq \operatorname{Max}_{C_1} = 0$

Repeat using u_2-u_1 instead. See $u_1 \equiv u_2 \leftarrow \Delta = 0$ ✓

Fact: Min princ for harmonic u follows from max princ for $-u$.