# Lecture 14 Fejer's Theorem

Thm: Suppose $f$ is piecewise continuous on $[-\pi,\pi]$ and differentiable at $x_0$. Then $S_N(x_0) \to f(x_0)$.

Also true if $f$ is Lipschitz at $x_0$: $|f(x)-f(x_0)| \le M|x-x_0|$

So also true at "kinks" in piecewise $C'$ fcns.

Weierstraß M-test: Suppose $\varphi_n(x)$ are continuous fcns on $[a,b]$ and $|\varphi_n(x)| \le M_n$ on $[a,b]$. If $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum \varphi_n$ converge uniformly on $[a,b]$ to a continuous fcn $f$.

Why: Fact: a uniform limit of continuous fcns is continuous.

Step 1: $\left\{\sum_1^N \varphi_n\right\}_{N=1}^{\infty}$ is uniformly Cauchy: Given $\varepsilon>0$,

there is an $N$ such that

$$\left|\sum_1^n \varphi_k(x) - \sum_1^m \varphi_k(x)\right| < \varepsilon \quad \text{for } x \in [a,b]$$

when $n$ and $m$ are both $> N$.

Why: $n > m$: $$\left|\sum_1^n \varphi_k - \sum_1^m \varphi_k\right| = \left|\sum_{m+1}^n \varphi_k\right|$$

$$\le \sum_{m+1}^n |\varphi_k| \le \sum_{m+1}^n M_k \le \underbrace{\sum_{m+1}^{\infty} M_k}_{\text{tail end of a conv series.}}$$

Can choose $N$ to make tail end as small as desired.

So $\sum_{1}^{N} \varphi_k$ is unif Cauchy.

Fact: Uniformly Cauchy sequences converge uniformly.

EX: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \underbrace{\sin(2n+1)x}_{\varphi_n}$ converges unif to a cont. fcn.

$$\left| \frac{(-1)^n}{n^2} \sin(2n+1)x \right| \leq \frac{1}{n^2} \leftarrow M_n \qquad \sum \frac{1}{n^2} < \infty \ \checkmark$$

Fact about inner products: Whenever $\langle u,v \rangle$ satisfies axioms of an inner product, get Cauchy-Schwarz ineq:

$$|\langle u,v \rangle| \leq \|u\| \|v\|.$$

EX: $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\|$ : $\displaystyle\left| \sum_{k=1}^{N} u_k \bar{v}_k \right| \leq \left( \sum_{1}^{N} |u_k|^2 \right)^{1/2} \left( \sum_{1}^{N} |v_k|^2 \right)^{1/2}$

$$\left| \int_a^b u(x) \overline{v(x)}\, dx \right| \leq \left( \int_a^b |u|^2 dx \right)^{1/2} \left( \int_a^b |v|^2 dx \right)^{1/2}$$

Homework prob: Fourier coeff for $f'$ satisfies

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$f' \sim \sum_{-\infty}^{\infty} (inc_n) e^{inx}$$

Why: Notation $\hat{f}(n) = c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$

$$\hat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)\; e^{-int}\; dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{e^{-int}}_{u}\; \underbrace{f'(t)\, dt}_{dv}$$

$$du = -in\, e^{-int} dt \qquad v = f(t)$$

$$\underset{\substack{\text{int by}\\ \text{parts}}}{=} \underbrace{\frac{1}{2\pi} uv\Big|_{-\pi}^{\pi}}_{\substack{=0\\ \text{when } f\\ 2\pi \text{ periodic}}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(t)}_{v}\; \underbrace{\left[-in\, e^{-int}\right] dt}_{du} = in\, c_n \; ✓$$

$$= in \cdot \hat{f}(n)$$

Bessel's ineq: Suppose $f$ is $C'$ $2\pi$ periodic

Then Bessel's: $\displaystyle\sum_{n=-\infty}^{\infty} |in c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'|^2\, dx \le M$

Hmmm: $$\sum |c_n| = \sum \frac{1}{n}\cdot |in c_n| \le \left(\sum \frac{1}{n^2}\right)^{1/2} \underbrace{\left(\sum |in c_n|^2\right)^{1/2}}_{\le M^{1/2}}$$

Can make tail end of $\sum |c_n|$ as small as desired. Weierstraß M-test $\Rightarrow$ Fourier series converges uniformly to a cont fcn. Combine with Thm at start:

**Thm:** If $f$ is $C^1$-smooth and $2\pi$-periodic, then F. Series converges to $f$ uniformly on $[-\pi, \pi]$.

Error estimate:

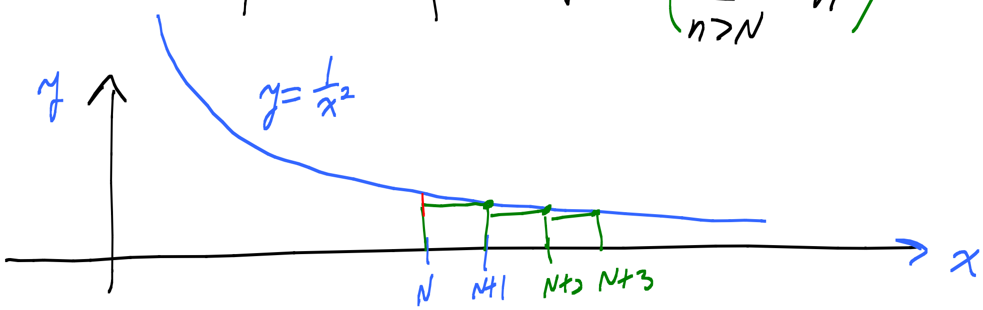
$$|f - S_N| = \Big| \sum_{|n|>N} c_n e^{inx} \Big|$$

$$\leq \sum_{|n|>N} |c_n| \cdot |n| \cdot \frac{1}{|n|}$$

$$\underset{\text{Cauchy-Schw}}{\leq} \sum |inc_n| \cdot \frac{1}{|n|} \leq \underbrace{\Big( \sum |nc_n|^2 \Big)^{1/2}}_{\leq M^{1/2}} \cdot \Big( 2 \sum_{n>N} \frac{1}{n^2} \Big)^{1/2}$$

Get $$|f - S_N| \leq \sqrt{2M} \Big( \sum_{n>N} \frac{1}{n^2} \Big)^{1/2} \leq \sqrt{2M} \Big( \int_N^\infty \frac{1}{x^2}\, dx \Big)^{1/2}$$

$$\leq \sqrt{\frac{2M}{N}}$$




Error estimate: $f$ is $C^1$ smooth and $2\pi$-periodic

$$|f - S_N| \leq \sqrt{2 \max_{[-\pi,\pi]} |f'|^2} \cdot \frac{1}{N^{1/2}}$$

Philosophy: The smoother $f$ is, the faster $S_N \to f$.

$$f \sim \sum c_n e^{inx}$$
$$f' \sim \sum in c_n e^{inx}$$
$$f'' \sim \sum -n^2 c_n e^{inx}$$

If $f$ is $C^2$-smooth: $\sum_{-\infty}^{\infty} n^4 |c_n|^2 < \infty$.

$$\underbrace{\widehat{f''}(n)} = -n^2 \hat{f}(n)$$

$$|\widehat{f''}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f'' e^{-int}\, dt \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f''| \underbrace{|e^{-int}|}_{=1} dt \le \frac{1}{2\pi} \max_{[-\pi,\pi]} |f''| \cdot 2\pi$$

$$|\widehat{f''}(n)| = n^2 |\hat{f}(n)|$$

Conclude $\underbrace{|\hat{f}(n)|}_{|c_n|} \le \frac{M}{n^2}$ where $M = \max_{[-\pi,\pi]} |f''|$.

Fejer's Thm: Take average of first $N$ Fourier partial sums:

$$\sigma_N(x) = \frac{S_1(x) + S_2(x) + S_3(x) + \dots + S_N(x)}{N} \leftarrow \text{Cesáro sums}$$

converges uniformly to $f$ when $f$ is just continuous and $2\pi$-periodic!

<u>Cor</u>: Trig polys are "dense" among $2\pi$ periodic continuous fcns.

<u>Cor</u>: Bessel's $\Rightarrow$ Parseval's when you know this.

<u>Suggestion</u>: Reread Seeley in order.