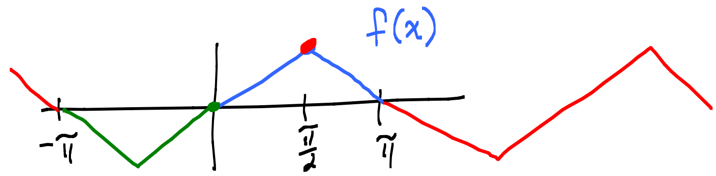
Lecture 15 Féjér's theorem

HWK 4 due Friday
Midterm in class
Monday, March 4

Remark: Harpsichord problem



Today we know $f(x) = \underbrace{\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots}$

Fourier sine series on $[0,\pi]$ is the full Fourier for the odd 2π-periodic extension.

converges pointwise to $f(x)$.

Aha! Weierstraß M-test using $M_n = \frac{1}{(2n-1)^2}$

[Note: $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + < \sum_1^\infty \frac{1}{n^2} < \infty$ by Int. Test.]

we conclude that the series converges uniformly to f.

Given a tolerance $\varepsilon > 0$, could figure out how many terms needed to get

$$|f - S_{(2n-1)}| = \left| \frac{\pm 1}{(2n+1)^2} \sin(2n+1)x + \cdots \right|$$

$$\leq \frac{1}{(2n+1)^2} + \frac{1}{(2n+3)^2} + \cdots$$

$$< \sum_{m=2n+1}^{\infty} \frac{1}{m^2} < \frac{1}{2n+1} < \frac{1}{100}$$

if $n \geq 50$

Cesáro means:

$$\sigma_N(x) = \frac{S_0(x) + S_1(x) + \cdots + S_{N-1}(x)}{N}$$

Féjér's Thm: If f is 2π-periodic and continuous, then $\sigma_N(x)$ converges uniformly to f on $\mathbb{R}$.

Why: Dirichlet kernel:

$$D_N(x) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi}\cdot\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

$$\boxed{S_N(x) = \int_{-\pi}^{\pi} D_N(t)\, f(x-t)\, dt}$$

$$\sigma_N(x) = \int_{-\pi}^{\pi} \underbrace{\frac{D_0(t) + D_1(t) + \cdots + D_{N-1}(t)}{N}}_{F_N(t) \,=\, \text{Féjér kernel}} f(x-t)\, dt$$

Fact: The Féjér kernel is a very good kernel:

1) $$F_N(t) = \frac{1}{2\pi N}\,\frac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)}$$

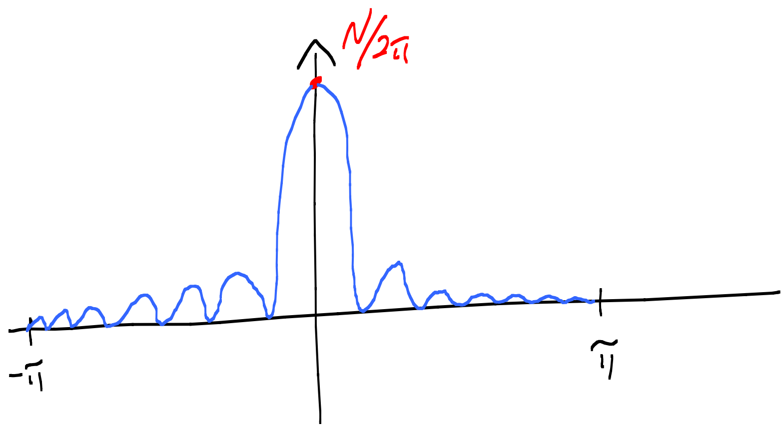
2) L'H $\Rightarrow$ $$F_N(0) = \frac{1}{2\pi N}\,\frac{\left(\frac{N}{2}\right)^2}{\left(\frac{1}{2}\right)^2} = \frac{N}{2\pi}$$ > 0 big!

3) $F_N(t) > 0$ for all t, even.

4) $\int_{-\pi}^{\pi} F_N(t)\,dt = 1$ (because $\int_{-\pi}^{\pi} D_m\,dt = 1$)

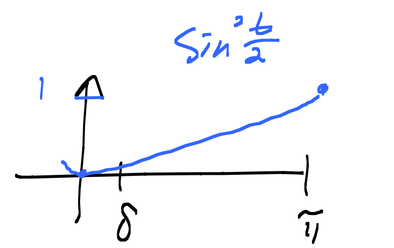
5) Let $I_\delta = [-\pi,-\delta] \cup [\delta,\pi]$ (punch out $(-\delta,\delta)$).

$F_N(t) \to 0$ uniformly on I_δ.



Pf: Easy from formula.

5). $$F_N(t) = \frac{1}{2\pi N} \frac{\sin^2 \frac{N}{2}t}{\sin^2 \frac{t}{2}}$$



$\sin^2 \frac{t}{2} \geq \sin^2 \frac{\delta}{2}$ on $[\delta, \pi]$.

$$0 < F_N(t) \leq \frac{1}{2\pi N} \cdot \frac{1}{\sin^2 \frac{\delta}{2}} \quad \text{on } I_\delta.$$

Pf of Féjer's:

$$\sigma_N(x) - f(x) = \int_{-\pi}^{\pi} F_N(t) f(x-t)\,dt - f(x) \overbrace{\int_{-\pi}^{\pi} F_N(t)\,dt}^{=1}$$

$$= \int_{-\pi}^{\pi} \underbrace{F_N(t)}\ \underbrace{\left[f(x-t) - f(x) \right]}_{\text{small near } t=0} \, dt$$

small away from $t=0$

Let $\varepsilon > 0$. $\exists\ \delta > 0$ such that $|f(x-t) - f(x)| < \frac{\varepsilon}{2}$ when $|t| < \delta$. [by unif cont.]

<u>Note:</u> Continuity on $[-\pi, \pi] \Rightarrow$ <u>uniform</u> continuity.

$$\text{Now } \left| \sigma_N(x) - f(x) \right| = \left| \underbrace{\int_{I_\delta} F_N(t) \left[f(x-t) - f(x) \right] dt}_{I_1} + \underbrace{\int_{-\delta}^{\delta} F_N(t) \left[f(x-t) - f(x) \right] dt}_{I_2} \right|$$

$$\leq |I_1| + |I_2|$$

$$|I_1| \leq \underbrace{\left(\max_{I_\delta} |F_N| \right)}\ 2M \cdot \underbrace{\text{Length}(I_\delta)}_{< 2\pi} < \frac{\varepsilon}{2}$$

$M = \max_{[-\pi, \pi]} |f|$

Can choose N big enough to make this $< \dfrac{\varepsilon/2}{4\pi M}$

$$|I_2| = \left| \int_{-\delta}^{\delta} F_N(t) \left[f(x-t) - f(x) \right] dt \right|$$

$$\leq \int_{-\delta}^{\delta} F_N(t) \left| f(x-t) - f(x) \right| dt$$

$F_N > 0$!

$|\ldots| < \frac{\varepsilon}{2}$

$$\leq \frac{\varepsilon}{2}\int_{-\delta}^{\delta} F_N(t)\,dt < \frac{\varepsilon}{2}\underbrace{\int_{-\pi}^{\pi} F_N(t)\,dt}_{=1} \leq \frac{\varepsilon}{2}$$

Cor: Trig polys are dense among continuous fcns (uniformly).

Cor: Parseval's identity.

Cor: Uniqueness of Fourier series: Suppose f and g are piecewise continuous. If $\hat{f}(n) = \hat{g}(n)$ for all n, then $f \equiv g$ on continuous parts.

And: If $\hat{f}(n) = 0$ for all n, then $f \equiv 0$.

Why: $\sum |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\,dt$ ← If $=0$, no continuous bumps where $f \neq 0$.