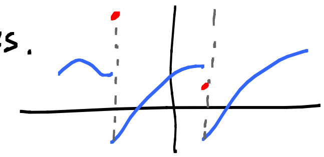
Lecture 16 Hot wire problem

HWK 5 on home page due next Friday
Midterm exam in class Mon, March 4

Fact: Suppose f is piecewise continuous.



$$\int_a^b |f|^2\,dx = 0 \implies f(x)=0 \text{ at points } x \text{ where } f \text{ is continuous.}$$

Me: Bessel's ineq $\implies$ Fourier coeff $\to 0$ as $n\to\infty$ for piecewise continuous fcns. (Seeley Chap 3)

Seeley in Chap 1: If f is C^1-smooth, then Fourier coeff $\to 0$ as $n\to\infty$.

Pf:

$$\int_a^b f(x) \underbrace{\sin nx}_{\substack{\text{or} \\ \cos nx}}\,dx = \int_a^b \underbrace{f(x)}_{u} \underbrace{\sin nx\,dx}_{dv}$$

$du = f'(x)\,dx$ $\quad$ $v=\int \sin nx\,dx = -\frac{1}{n}\cos nx$

$$= uv\Big|_a^b - \int_a^b v\,du$$

$$= \left[-\frac{1}{n} f(x)\cos nx\right]_a^b + \frac{1}{n}\underbrace{\int_a^b f'(x)\cos nx\,dx}_{\left|\int_a^b\right| \le \left(\max_{[a,b]}|f'|\right)\cdot 1\cdot(b-a)}$$

$\to 0$ as $n\to\infty$.

Riemann-Lebesgue lemma: If $f\in L^1[-\pi,\pi]$ $\leftarrow$ f is measurable and $\int_{-\pi}^{\pi}|f|\,dx<\infty$

Then Fourier coeff $\to 0$ as $n\to\infty$.

Féjér kernel

$$F_N(x) = \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N}$$

where $D_m(x) = \sum_{n=-m}^{m} e^{imx}$

$$F_N(x) = \frac{Ne^0 + (N-1)\left[e^{-ix}+e^{ix}\right] + (N-2)\left[e^{-2ix}+e^{2ix}\right] + \cdots + (1)\left[e^{-i(N-1)x}+e^{i(N-1)x}\right]}{N}$$

$$= D_{N-1}(x) - \frac{1}{N}\left[\left(e^{-ix}+e^{ix}\right) + 2\left(e^{-i2x}+e^{i2x}\right) + \cdots + (N-1)\left(e^{-i(N-1)x} + e^{i(N-1)x}\right)\right]$$

$2e^{i2x} = 2(e^{ix})^2$

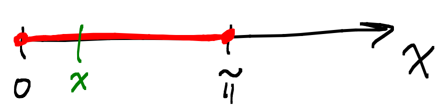
Hmmm: $1 + w + w^2 + \cdots + w^N = \frac{1-w^{N+1}}{1-w}$

Aha! Differentiate: $0 + 1 + 2w + \cdots + Nw^{N-1} = \frac{d}{dw}\left(\frac{1-w^{N+1}}{1-w}\right)$

Multiply by w: $1\cdot w + 2w^2 + \cdots + Nw^n = w\frac{d}{dw}\left(\frac{1-w^{N+1}}{1-w}\right)$

Let $w = e^{ix}$. Then let $w = e^{-ix}$. Algebra!

Hot wire problem



$u(x,t)$ = temp of wire at point x at time t.

Heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

(Homogeneous) Boundary conditions: Hold endpoints at <u>zero</u> degrees:

$$u(0,t)=0\ ,\qquad u(\pi,t)=0$$

Initial condition $\quad u(x,0)=f(x)$ ← know f

<u>Derive heat equation:</u>

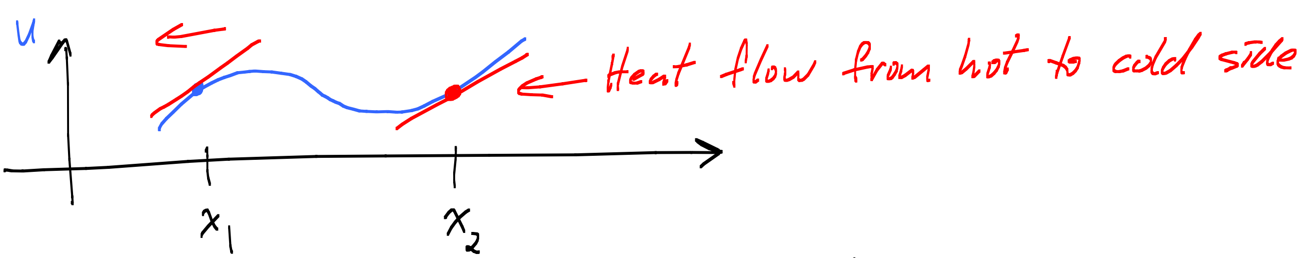
Total heat content in wire between x_1 and x_2 is

$$c\int_{x_1}^{x_2} u(x,t)\,dx$$

Rate changing: $\quad \dfrac{d}{dt}\left(c\displaystyle\int_{x_1}^{x_2} u(x,t)\,dx\right)$

$$= c\int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx$$

But: Changing only because of heat flow at x_1 or x_2



Kelvin's law: (rate of heat flow) $=$ (negative constant) $\dfrac{\partial u}{\partial x}$

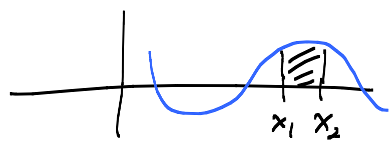
Conservation of energy:

$$\int_{x_1}^{x_2} c\,\frac{\partial u}{\partial t}(x,t)\,dx = k\,\frac{\partial u}{\partial x}(x_2,t) - k\,\frac{\partial u}{\partial x}(x_1,t)$$

$$= k\int_{x_1}^{x_2} \frac{\partial^2 u}{\partial x^2}(x,t)\,dx \quad \leftarrow \text{by Fund Thm Calc.}$$

$$\int_{x_1}^{x_2} \left[c\,\frac{\partial u}{\partial t}(x,t) - k\,\frac{\partial^2 u}{\partial x^2}(x,t) \right] dx = 0$$

Aha! True for any $x_1 < x_2$.



Must be that integrand is $\equiv 0$. Heat equ ✓

Method of separation of variables: Look for sol'ns of form

$$u(x,t) = X(x)T(t).$$ (Take $c=1$.)

PDE.
$$\frac{\partial}{\partial t}\left[X(x)T(t)\right] \overset{\text{want}}{=} \frac{\partial^2}{\partial x^2}\left[X(x)T(t)\right]$$

$$XT' = X''T$$

$$\frac{X''}{X} = \frac{T'}{T} \overset{\text{must = a const!}}{=} \lambda$$

Get two ode's :

$$X'' - \lambda X = 0$$

Boundary Cond.

$u(0,t) = X(0)T(t) = 0$ want

$u(\pi,t) = X(\pi)T(t) = 0$

Don't want $T(t) \equiv 0$.

Need

$$\boxed{\begin{aligned} X(0)&=0 \\ X(\pi)&=0 \end{aligned}}$$

BC

Famous Sturm-Liouville problem

Only get sol'ns not $\equiv 0$ when

$\lambda = -n^2$

$X_n(x) = \sin nx$

$$T' - \lambda T = 0$$

Back to

T prob :

$$T' - \underbrace{(-n^2)}_{\lambda} T = 0$$

$$\frac{dT}{dt} = -n^2 T$$

exponential decay

Sol'n $T_n(t) = Ce^{-n^2 t}$