Lecture 17 Hot wire problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad BC \begin{cases} u(0,t)=0 \\ u(\pi,t)=0 \end{cases} \qquad IC \quad u(x,0)=f(x)$$

Try $u(x,t) = X(x)\,T(t)$. Got

$$X'' - \lambda X = 0 \qquad\qquad T' - \lambda T = 0$$

$$\begin{cases} X(0)=0 \\ X(\pi)=0 \end{cases}$$

For $\lambda = -n^2$:

$$T' = -n^2 T$$

$\lambda = -n^2$. Got

$$X_n(x) = \sin nx$$

Got $T_n(t) = e^{-n^2 t}$

and so $u_n(x,t) = e^{-n^2 t} \sin nx$

Hope $$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx$$

is a sol^n and we can make

want

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = f(x)$$

Yes! Fourier Sine series: $A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$.

EX: Take $f(x) \equiv 1$.

$$A_n = \frac{2}{\pi}\int_0^{\pi} 1 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

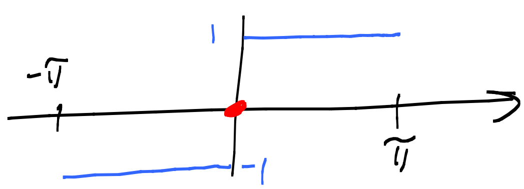
$$= \frac{2}{\pi n}\left[-\cos n\pi - (-\cos 0)\right]$$

$$= \frac{2}{\pi n}\left(1-(-1)^n\right) = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$u(x,t) = \frac{4}{\pi}\Big(e^{-t}\sin x + \frac{1}{3}e^{-3t}\sin 3x + \frac{1}{5}e^{-5t}\sin 5x + \frac{1}{7}e^{-7t}\sin 7x + \cdots\Big)$$

Great things about Fourier series $\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots$

1) All sine fcns are odd. This series converges to odd 2π-periodic extension



2) Parseval's : $$\underbrace{\sum_{-\infty}^{\infty}|c_n|^2}_{|a_0|^2 + 2\sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2\,dx$$

$$0 + 2\sum_{n=1}^{\infty}\left(0^2 + \frac{(4/\pi)^2}{(2n-1)^2}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx$$

$$2\left(\frac{4}{\pi}\right)^2\left(\underbrace{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}_{S}\right)=1$$

$$S=\frac{\pi^2}{32} \quad !$$

Let $x=\frac{\pi}{2}$:

$$1=\frac{4}{\pi}\left(\underbrace{\frac{1}{1}\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\cdots}_{\underbrace{\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots}_{S_2}}\right)$$

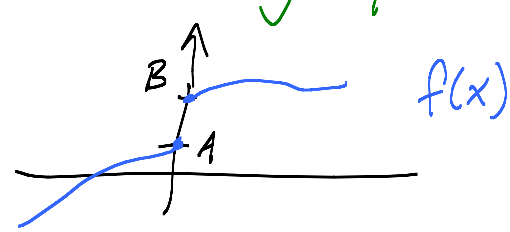
Get $S_2=\frac{\pi}{4}$

Freshman calc
$$\underbrace{\int_0^1 \frac{1}{1+x^2}\,dx}_{\substack{\arctan x\big|_0^1 \\ \| \\ \frac{\pi}{4}}} = \int_0^1 \frac{1}{1-(-x^2)}\,dx$$

$$= \int_0^1 1-x^2+x^4+\cdots+(-1)^n x^{2n}\,dx$$

$$+\int_0^1 \underbrace{\frac{(-1)^{n+1}x^{2(n+1)}}{1+x^2}}_{\substack{\text{error}\\ \text{in geom series}\\ \text{formula}}}\,dx$$

Jumps! Suppose f is piecewise C^1 with a jump from A to B at $x=0$:



Let $H(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ $\quad \frac{1}{2}\left(H(x)+1\right) \leftarrow$ Jumps from 0 to 1 at $x=0$

$$f(x) - \frac{(B-A)}{2}\left(H(x)+1\right) \text{ has no jump at } x=0!$$

It might have a "kink" at $x=0$, but that's ok.

Fourier series for a sum is the sum of F.S.

Conclude: F.S. for f converges to $\frac{A+B}{2}$ at $x=0$, the midpoint of the jump.

Next time: Hot wire with insulated ends.

Kelvin: $(\text{Rate of heat flow}) = -k\left(\underbrace{\text{Temp. gradient}}_{\frac{\partial u}{\partial x}}\right)$

So $\begin{cases} \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(\pi,t) = 0 \end{cases}$

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\partial u}{\partial x} = X'(x)\,T(t)$$

BC $\begin{cases} X'(0)\,T(t) = 0 \\ X'(\pi)\,\underbrace{T(t)} = 0 \end{cases}$

don't want $T(t) \equiv 0$

New Sturm-Liouville problem for X:

$$X'' - \lambda X = 0$$

$$\begin{cases} X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$