Lecture 18 Hot wire – insulated ends

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

BC $\begin{cases} \dfrac{\partial u}{\partial x}(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(\pi,t) = 0 \end{cases}$ IC $u(x,0) = f(x)$

$$X'' - \lambda X = 0 \qquad\qquad T' - \lambda T = 0$$

BC $\begin{cases} X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$

Case $\lambda = 0$: $X'' = 0$

$$X(x) = Ax + B$$

BC $\Rightarrow$ $A = 0$. Nothing about B!

For $\lambda = 0$, get non-zero sol^n $X_0(x) = 1$

Case $\lambda > 0$: $\lambda = k^2$ $X'' - k^2 X = 0$

$$r^2 - k^2 = 0$$

$$r = \pm k$$

$$X = Ae^{kx} + Be^{-kx} = \cancel{a \underbrace{\sinh kx}_{\frac{e^{kx} - e^{-kx}}{2}}} + b \underbrace{\cosh kx}_{\frac{e^{kx} + e^{-kx}}{2}}$$

BC: $X'(x) = \cancel{ak \cosh kx} + bk \sinh kx$

$$X'(0) = ak \cdot 1 + bk \cdot 0 = ak \overset{\text{want}}{=} 0 \qquad \underline{a=0}.$$

$$X'(\pi) = bk \underbrace{\sinh k\pi}_{\neq 0} \overset{\text{want}}{=} 0 \qquad \underline{b=0} \text{ too!}$$

Only get $X(x) \equiv 0$ sol^n.

<u>Case $\lambda < 0$</u>: $\lambda = -k^2$

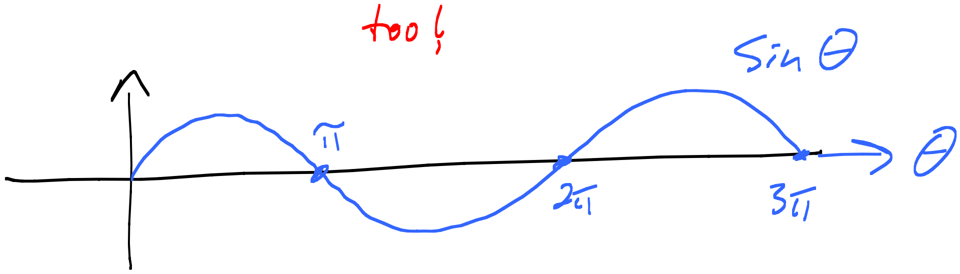
$$X'' + k^2 X = 0$$
$$r^2 + k^2 = 0$$
$$r = \pm k i$$

$$X(x) = A\cos kx + \cancel{B\sin kx}$$
$$X'(x) = -Ak\sin kx + \cancel{Bk\cos kx}$$

BC:
$$X'(0) = -Ak \cdot 0 + Bk \cdot 1 \overset{\text{want}}{=} 0 \qquad \underline{\underline{B=0}}$$
$$X'(\pi) = -Ak \underbrace{\sin k\pi}_{} \overset{\text{want}}{=} 0$$

Don't want $A=0$ too!

need $= 0$



Need $k\pi = n\pi$, $n = 1, 2, 3, \dots$ $\qquad \boxed{k = 1, 2, 3, \dots}$

For $\lambda_n = -n^2$. Get $X_n(x) = \cos nx$ (Take $A = 1$)

Back to $T(t)$: $\quad T' - \lambda T = 0$

For $\lambda_n = -n^2$: $\quad T' = -n^2 T$

$\lambda = 0$: $T' = 0$
$T \equiv$ const c.
Take $c = 1$

$$T_n(t) = ce^{-n^2 t} \quad \left(\text{Take } c = 1\right)$$

Get $u_n(x,t) = X_n(x) T_n(t) = e^{-n^2 t} \cos nx$

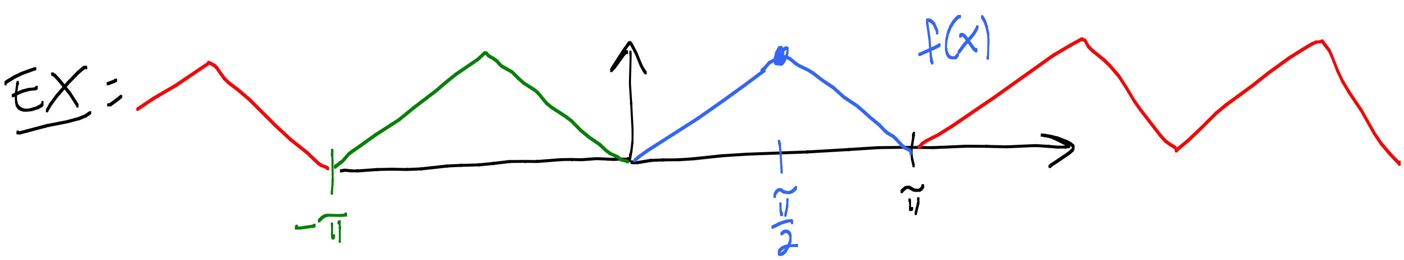
Hope can get $u(x,t)$ that solves IC via

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2 t} \cos nx}$$

IC $\quad u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx$

Fourier Cosine series!

EX:



Even extension. $\quad 2\pi$-periodic ext.

Full Fourier series for even 2π-periodic ext

Sine terms: $\quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\sin nx}_{\text{odd}}}_{\text{odd}} \, dx = 0$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}}\,dx = \frac{2}{2\pi}\int_0^{\pi} f(x)\,dx$$

$$= \underset{\text{on } [0,\pi]}{\text{Ave of } f}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}}\ \underbrace{\cos nx}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$u(x,t) = \ldots$ with $A_0 = \underset{\text{on } [0,\pi]}{\text{Ave of } f}$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Fourier Cosine Series coeff

<u>Interesting fact</u>: $\lim_{t\to\infty} u(x,t) =$

$$\lim_{t\to\infty}\left[A_0 + \sum_{n=1}^{\infty} A_n \underbrace{e^{-n^2 t}}_{\to 0 \text{ as } t\to\infty} \cos nx\right] = A_0$$

Aha! Steady state (or equillibrium) temp

= Ave initial temp.

<u>Homework-like prob</u>:

$$X'' - \lambda X = 0$$

$$\begin{cases} X'(0) = 0 & \leftarrow \text{insulated at } x=0 \\ X(\pi) = 0 & \leftarrow \text{Iced at } x=\pi \end{cases}$$

Look for "eigenvalues" λ and "eigenfunctions" $X \leftarrow$ not $\equiv 0$.

Claim: Sol^{n}s are automatically $\perp$ on $[0,\pi]$!

Eigen: $X'' = \lambda X$ $\quad L = \frac{d^2}{dx^2} \leftarrow$ linear operator

λ_1, X_1 and λ_2, X_2 $\quad \lambda_1 \neq \lambda_2$

$$\lambda_1 \langle X_1, X_2 \rangle = \langle \underbrace{\lambda_1 X_1}_{X_1''}, X_2 \rangle$$

$$\int_0^{\pi} X_1'' X_2 \, dx = \int_0^{\pi} \underbrace{X_2}_{u} \underbrace{X_1'' \, dx}_{dv}$$

$$= \underbrace{X_2 X_1' \Big|_0^{\pi}}_{=0 \text{ by BCs}} - \int_0^{\pi} \underbrace{X_2'}_{u} \underbrace{X_1' \, dx}_{dv}$$

$$= \underbrace{X_2' X_1 \Big|_0^{\pi}}_{=0 \text{ by BCs}} + \int_0^{\pi} X_2'' X_1 \, dx$$

$$\lambda_1 \langle X_1, X_2 \rangle = \langle X_1, \underbrace{X_2''}_{\lambda_2 X_2} \rangle = \lambda_2 \langle X_1, X_2 \rangle$$

Subtract $\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{\langle X_1, X_2 \rangle}_{\substack{\text{eigenfunctions} \\ \text{of different e.vals} \\ \text{must be } \perp.}} = 0$

Generalized Fourier series:

$$f(x) \overset{\text{want}}{=} \sum_{n=1}^{\infty} c_n X_n(x)$$

Get $\langle f, X_m \rangle = c_m \langle X_m, X_m \rangle$

$$c_m = \frac{\langle f, X_m \rangle}{\langle X_m, X_m \rangle}$$