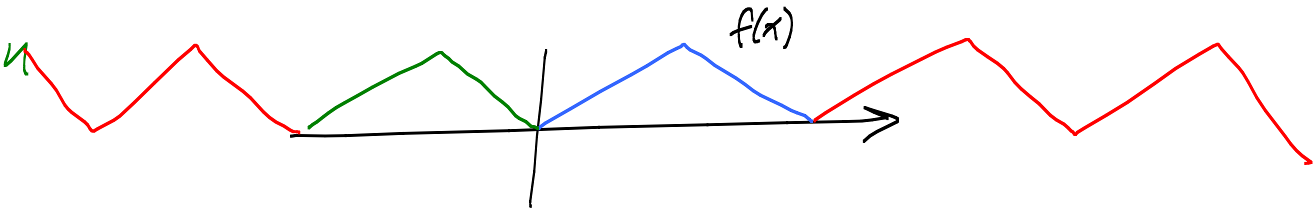
Lecture 19

Last time: viewed cosine series via even 2π-periodic extension



No jumps. Piecewise C^1-smooth. Cosine series (= Full Fourier series for even 2π-per. ext) converges to $f(x)$. Maple calculation shows $c_n \sim \frac{1}{n^2}$.

Weierstraß M-test shows F. Series converges uniformly.

Alternate way: $1, \cos nx$ are $\perp$ on $[0,\pi]$.

Can expand on $[0,\pi]$ using this $\perp$ basis for L^2.

Fact: If the Fourier series for f converges uniformly to g, then g is continuous.

If f is continuous, then $f \equiv g$.

Bessel's, Parseval's, Féjér's

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2 - \sum_{n=-N}^{N}|c_n|^2 = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f-S_N|^2\,dx}_{\geq 0} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f-T_N|^2\,dx$$

Obvious fact: S_N can be viewed as trig poly of degree $M > N$ be setting the extra coeff $=0$.

$$\text{So} \quad \frac{1}{2\pi}\int |f - S_{N+1}|^2\,dx \leq \frac{1}{2\pi}\int |f - S_N|^2\,dx$$

Use Féjér's: Suppose f is 2π-periodic and continuous. Let $\varepsilon > 0$. Féjér's $\Rightarrow$ there is a Cesàro ave

$$T_{N_0}(x) = \underbrace{\frac{S_0(x) + S_1(x) + \cdots + S_{N_0-1}(x)}{N_0}}_{\text{Trig poly degree } N_0-1.}$$

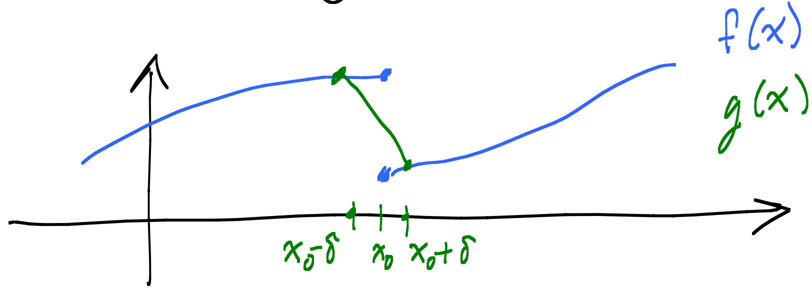
Max $|f(x) - T_{N_0}(x)| < \varepsilon$ on $[-\pi, \pi]$. Let $N > N_0 - 1$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - S_N|^2\,dx \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f - S_{N_0-1}|^2\,dx \leq \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - T_{N_0}|^2\,dx}_{\leq \frac{1}{2\pi}\cdot\varepsilon^2\cdot 2\pi \,=\, \varepsilon^2}$$

Conclude that $\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f - S_N|^2\,dx}_{0 \,\leq\, \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2 - \sum_{n=-N}^{N}|c_n|^2} \to 0$ as $N \to \infty$ $\leftarrow$ squeezing down to 0.

Parseval's from Féjér's for continuous f. ✓

What if f has jumps?



same as f, but takes connector line from $x_0-\delta$ to $x_0+\delta$.

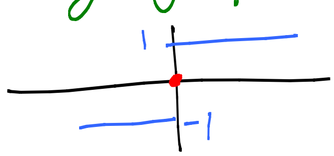
$$\int |f-g|^2\,dx = \int_{x_0-\delta}^{x_0+\delta} |\underbrace{\text{blip}}_{<\text{ jump dist } M+1}|^2\,dx < (M+1)^2\, 2\delta$$

Aha! If f is piecewise cont., can find continuous g so that $\int |f-g|^2\,dx < \varepsilon$.

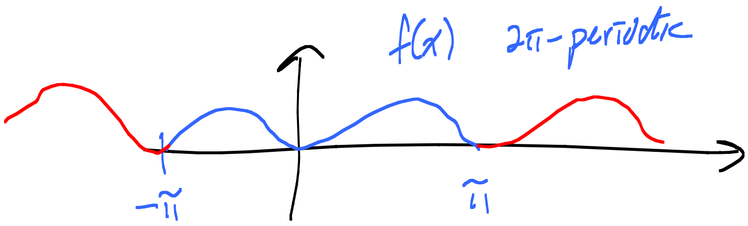
Get Féjer poly T_N close to g.

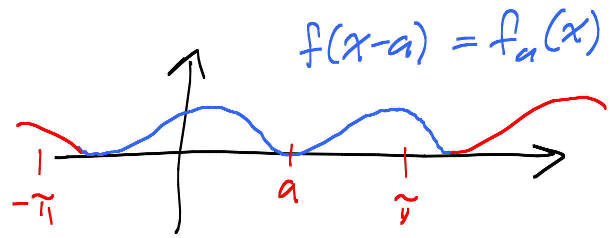
$$\|f - T_N\| = \|f - g + g - T_N\|$$
$$\le \underbrace{\|f-g\|}_{\text{small}} + \underbrace{\|g-T_N\|}_{\text{small}}$$

Can squeeze $\|f-S_N\|^2 \le \|f-T_N\|^2$ as above get Parseval's for piecewise continuous f.

Fact: Understanding jump behavior of F. series S_N for  at $x=0$ is all we need to know to understand jumps any place.

Why: (Fourier series of shifted f) = (shifted Fourier series for f)





$$\hat{f}_a(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_a(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\underbrace{x-a}_{\tilde{\tau}})\, e^{-inx}\, dx$$

$\tilde{\tau} = x - a \qquad x = \tilde{\tau} + a$

$d\tilde{\tau} = dx$

when $x = -\pi$: $\tilde{\tau} = -\pi - a$

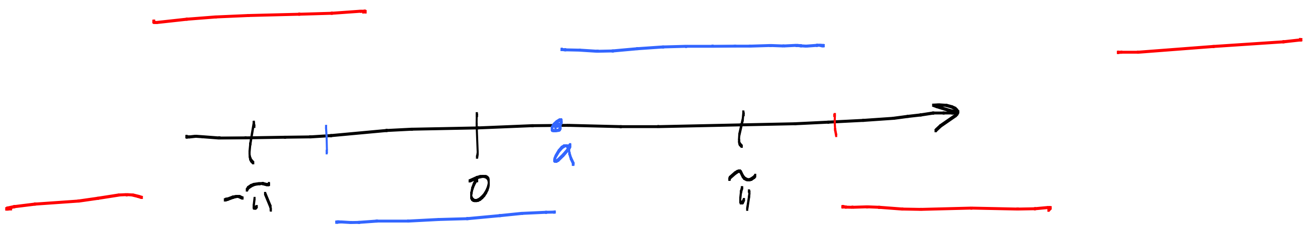
$x = \pi$: $\tilde{\tau} = \pi - a$

$$= \frac{1}{2\pi}\int_{-\pi-a}^{\pi-a} f(\tilde{\tau})\, \underbrace{e^{-in(\tilde{\tau}+a)}}_{e^{-in\tilde{\tau}}\, e^{-ina}}\, d\tilde{\tau}$$

$$= e^{-ina} \cdot \underbrace{\frac{1}{2\pi}\int_{\underbrace{-\pi-a}_{\text{interval length } 2\pi}}^{\pi-a} \underbrace{f(\tilde{\tau})\, e^{-in\tilde{\tau}}}_{2\pi\text{-periodic}}\, d\tilde{\tau}}_{= \int_{-\pi}^{\pi}}$$

So $\boxed{\hat{f}_a(n) = e^{-ina}\, \hat{f}(n)}$

$$S_N^{f_a} = \sum_{-N}^{N} \underbrace{c_n}_{\text{for } f} e^{-ina} e^{inx} = \underbrace{\sum_{-N}^{N} c_n e^{in(x-a)}}_{S_N^f(x-a)}$$



Thm: $S_N \to$ midpoint of jumps.