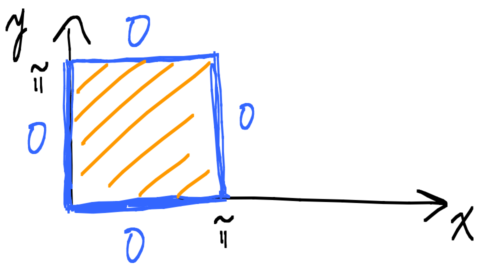
Lecture 20 Square drum



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

want

$$u_{tt} = X Y T'' = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) X Y T$$

$$= c^2 \left(X'' Y T + X Y'' T\right)$$

Divide by $c^2 X Y T$: $\dfrac{1}{c^2}\dfrac{T''}{T} = \dfrac{X''}{X} + \dfrac{Y''}{Y} = \lambda$

must be a constant.

[First let $t = t_0$, fixed. See the RHS = const (call it λ).
Now let t vary and fix $x = x_0$, $y = y_0$. LHS = same const.]

Save T prob $T'' - c^2 \lambda T = 0$ for last.

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

Aha! $\dfrac{X''}{X} = \lambda - \dfrac{Y''}{Y} = \mu$, a const.

fcn of x = fcn of y

Two problems:

$$X'' - \mu X = 0 \qquad \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

$$Y'' - (\lambda - \mu) Y = 0 \qquad \begin{cases} Y(0) = 0 \\ Y(\pi) = 0 \end{cases}$$

want

Remark: $u(\pi, y, t) = \underbrace{X(\pi)}_{\text{need } = 0} \underbrace{Y(y)}\, \underbrace{T(t)} = 0$

don't want $\equiv 0$ sol^n

X prob: Only get non-zero sol^{n}s if $\mu = -n^2$, $n = 1, 2, 3, \ldots$

Get $X_n(x) = \sin nx$

Y prob: $Y'' - \underbrace{(\lambda - (-n^2))}_{-m^2} Y = 0 \qquad \begin{cases} Y(0) = 0 \\ Y(\pi) = 0 \end{cases}$

only get non $\equiv$ zero sol^{n}s, $m = 1, 2, 3, \ldots$

$$Y_m(y) = \sin my$$

Back to T-prob. $\quad \mu = -n^2 \qquad \lambda - (-n^2) = -m^2$

$$\boxed{\lambda = -(n^2 + m^2)}$$

$$T'' + c^2(n^2 + m^2) T = 0$$

$$T_{n,m}(t) = A_{nm} \underbrace{\cos c\sqrt{n^2 + m^2}\, t}_{\text{for position at } t=0} + B_{nm} \underbrace{\sin c\sqrt{n^2 + m^2}\, t}_{\text{for velocity at } t=0}$$

Get $u_{n,m}(x,y,t) = X_n(x)\, Y_m(y)\, T_{n,m}(t)$

$$u(x,y,t) = \sum_{n,m=1}^{\infty} \Big(A_{nm} \sin nx \sin my \cos c\sqrt{n^2+m^2}\,t + B_{nm} \sin nx \sin my \sin c\sqrt{n^2+m^2}\,t \Big)$$

Initial conditions:

$$\begin{cases} u(x,y,0) = f(x,y) \leftarrow \text{initial shape} \\ \frac{\partial u}{\partial t}(x,y,0) = g(x,y) \leftarrow \text{initial speed (from drum stick hit)} \end{cases}$$

New problem: $u(x,y,0) \overset{\text{want}}{=} f(x,y)$

$$(*) \quad \boxed{\sum_{n,m=1}^{\infty} A_{nm} \sin nx \sin my = f(x,y)}$$

Double Fourier sine series!

One method. Fix y. Get F. Sine series in x.

Now fix x, . . .

or realize $\sin nx \sin my$ are $\perp$ via

$$\langle u, v \rangle = \int_0^{\pi}\int_0^{\pi} u(x,y)\, v(x,y)\, dx\, dy$$

Aha! Multiply $(*)$ by $\sin Nx \sin My$ and $\iint$.

$$A_{NM} \int_0^{\pi}\int_0^{\pi} \sin^2 Nx \sin^2 My \, dx\, dy = \int_0^{\pi}\int_0^{\pi} f(x,y) \sin Nx \sin My \, dx\, dy$$

$$\frac{\pi}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}$$

$$A_{NM}=\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y)\,\sin Nx\,\sin My\,dx\,dy$$

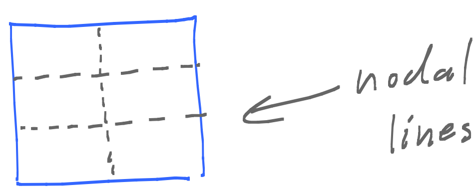
... etc.

Overtones of a square drum: frequencies $c\sqrt{n^2+m^2}$

Discordant!

Pitch of a drum = lowest freq: $c\sqrt{1^2+1^2}=c\sqrt{2}$

Modes of oscillation: $\underbrace{\sin 2x\,\sin 3y}_{\substack{\Delta u=-(2^2+3^2)u \\ \text{eigenfcn for } \Delta}}\cos c\sqrt{2^2+3^2}\,t$



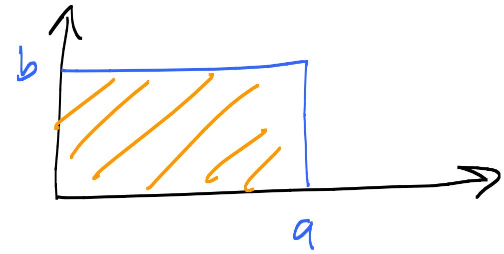
$65=n^2+m^2$ in <u>more</u> than two ways.

$$\Big(\underbrace{A\sin n_1x\,\sin m_1y+B\sin n_2x\,\sin m_2y}_{\text{interesting nodal curves!}}\Big)\cos c\sqrt{\ }\,t$$

Famous problem: Can you hear the shape of a drum?

Answer: No!

Rectangular drum:



$$u(x,y,t) = \sum_{n,m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{a} \cdot \sin\frac{m\pi y}{b} \cos c \quad t$$

$$+ \sum B_{nm}$$

Prob: Among rectangular drums of area A, which one is lowest?

Ans: The square one!

$A = ab \qquad f = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$