Lecture 21 Vibrating circular membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u = c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right]$$

BC $u(1, \theta, t) \equiv 0$

IC $\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{cases}$

Try $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

Get usual T prob: $T'' + \underset{\uparrow c>0}{(c)}\, T = 0$

For Θ: also get $\Theta'' + \underset{\uparrow c>0}{(c)}\, \Theta = 0$

with periodic boundary conditions.

Get full fourier series in Θ.

For $R(r)$: Get Bessel fcns!

Simplify prob. Look for sol'ns that are radially symmetric.

$$u(r,t) = R(r)T(t)$$

Get R problem $\boxed{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \lambda} = \frac{T''}{T}$

Case $\lambda = 0$: $r^2 R'' + r R' = 0$ Euler equ

Try $R(r) = x^p$:

$$p(p-1) + p = 0$$
$$p^2 = 0$$

$R_1(r) = 1$. $R_2(r) = \underbrace{1 \cdot \ln r}$

ouch! Rips hole in drum. This one not allowed.

Boundary Condition $u(1,t) = \underbrace{R(1)}_{\text{need } = 0.} \cdot T(t)$

No non-zero sol^n for $\lambda = 0$.

Case $\lambda > 0$: Can show no non-zero bounded sol^{n}s.

Case $\lambda < 0$: Bessel's equ. Bessel fcns.

ODE $$r^2 R'' + r R' - \lambda r^2 R = 0$$

$\lambda = -k^2$

Make a change of vars: $x = kr$.

ODE becomes: $$x^2 R'' + x R' + x^2 R = 0$$

Step 1: Use power series to find a sol^n.

$$x^2\left[R(x) = \sum_{n=0}^{\infty} a_n x^n\right] = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$x\left[R'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}\right] = \sum_{n=1}^{\infty} n a_n x^n$$

$$x^2\left[R''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right] = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$\underbrace{a_1}_{\text{must } = 0} x + \sum_{n=2}^{\infty} \underbrace{\left[n(n-1)a_n + n a_n + a_{n-2}\right]}_{\text{must } = 0} x^n \equiv 0$$

$$\boxed{a_n = \frac{-a_{n-2}}{n^2}} \qquad n = 2, 3, \ldots$$

recursion formula

$y(0) = a_0$ $\qquad a_1 = 0,\ a_3 = 0,\ a_5 = 0, \ldots$

$\underline{n=2}$: $\quad a_2 = \dfrac{-a_0}{2^2}$

$\underline{n=4}$: $\quad a_4 = \dfrac{-a_2}{4^2} = \dfrac{a_0}{4^2 \cdot 2^2}$

$\underline{n=6}$: $\quad a_6 = \dfrac{-a_4}{6^2} = \dfrac{-a_0}{6^2 \cdot 4^2 \cdot 2^2}$

$\vdots$

$$a_{2n} = \frac{(-1)^n}{2^{2n}(n!)^2} a_0$$

$$2^2 \cdot 4^2 \cdots (2n)^2 = (1\cdot 2)^2 (2\cdot 2)^2 \cdots (n \cdot 2)^2$$

Take $a_0 = 1$.

Bessel fcn of order zero

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

Can use "reduction of order" to get a second sol^n of form $v(x) J_0(x)$.

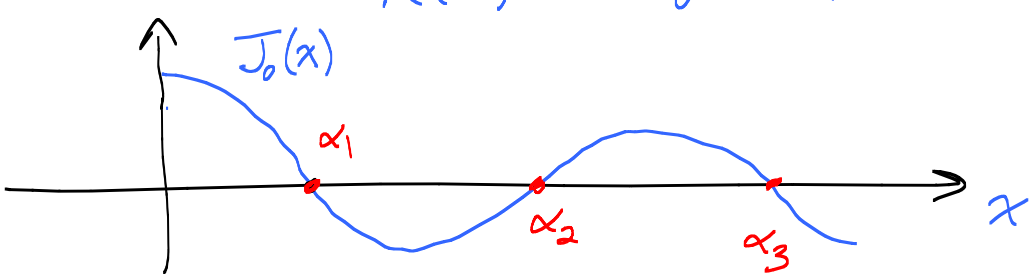
Plug into ODE. Get first order ODE in v.

Solve it. Ouch! v blows up at $x=0$.

Second sol^n is a drum ripper. Not allowed.

So $\quad R(x) = J_0(x)$

$$R(r) = J_0(kr)$$



BC $\quad R(1) = J_0(k \cdot 1) = 0$

Aha! Need k's to be zeroes of J_0! $\quad \lambda = -k^2$

Sol^n $\quad u(r,t) = \sum_{n=1}^{\infty} \left(A_n J_0(\alpha_n r) \cos \alpha_n t + B_n J_0(\alpha_n r) \sin \alpha_n t \right)$

IC $\quad u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \overset{\text{want}}{=} f(r)$

$$\frac{\partial u}{\partial t}(r,0) = \sum \alpha_n B_n J_0(\alpha_n r) = g(r)$$

Theory of singular Sturm-Liouville prob.

$\varphi_n(r) = J_0(\alpha_n r)$ are orthogonal

on $[0,1]$ with respect to inner product

$$\langle \varphi_n, \varphi_m \rangle = \int_0^1 \varphi_n(r)\, \varphi_m(r)\ \underset{\substack{\uparrow \\ \text{weight fcn.}}}{r}\ dr$$