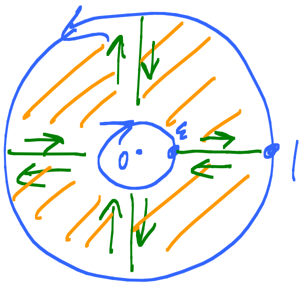
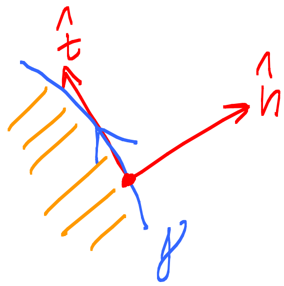
Lecture 23 Getting ready for Isoparametric Ineq

Averaging property of harmonic fcns



$$\left(\int_{C_1} - \int_{C_\varepsilon}\right) P\,dx - Q\,dy = \iint_\Omega \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\,dx\,dy$$



Standard orientation

$\hat{n}$ is $\hat{t}$ rotated 90° clockwise

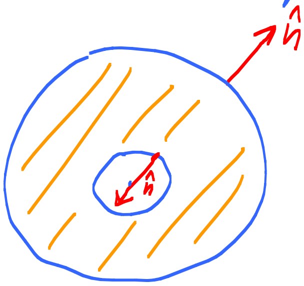
$\mathbb{C}$: $\hat{t} \sim$ complex # $= \dot{x} + i\dot{y}$

Rotate $\hat{t}$ by θ clockwise : $e^{-i\theta}\,\hat{t}$

$$\hat{n} \sim \underbrace{e^{-i\pi/2}}_{0 - i\cdot 1} \cdot (\dot{x} + i\dot{y}) = \dot{y} - i\dot{x}$$

So $\hat{n} = \dot{y}\,\hat{\imath} - \dot{x}\,\hat{\jmath}$

Green's $\longrightarrow$ Divergence Thm :
$$\int_\gamma \vec{F}\cdot\hat{n}\,ds = \iint_\Omega \operatorname{Div}\vec{F}\,dx\,dy$$



Outward pointing normal vectors

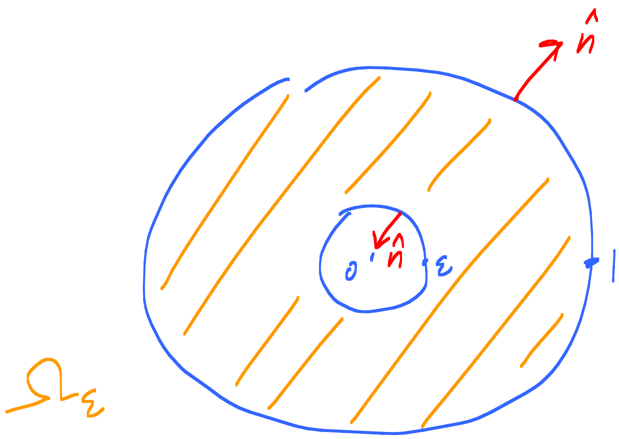
Ave property: Need $\ln r = \ln\sqrt{x^2+y^2}$

[$\ln r$ is the "fundamental sol^n to the Laplace operator:"

$$\frac{1}{2\pi}\Delta(\ln r) = \delta \quad \leftarrow \text{delta fcn at } 0$$

$$\iint_\Omega \frac{1}{2\pi}\Delta(\ln r)\,\varphi\, dA = \varphi(0) = \iint \delta\,\varphi\, dA$$
]

In $\mathbb{R}^3$: $\frac{1}{r}$ takes the place of $\ln r$



Green's #2

$$\int_\gamma u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, ds = \iint_{\Omega_\varepsilon} \underbrace{u\Delta v - v\Delta u}_{\equiv 0}\, dA$$

u = given harmonic fcn

$v = \ln r$, also harmonic

Aha! Radial direction is the normal direction.

$$\text{On } C_1: \quad \frac{\partial v}{\partial n} = \frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\ln r\Big|_{r=1} = \frac{1}{r}\Big|_{r=1} = 1$$

$$\text{On } C_\varepsilon: \quad \frac{\partial v}{\partial n} = -\frac{\partial v}{\partial r} = -\frac{\partial}{\partial r}\ln r\Big|_{r=\varepsilon} = -\frac{1}{\varepsilon}$$

Green's #2

Outer:
$$\int_{C_1} u \underbrace{\frac{\partial v}{\partial r}}_{=1} - \underbrace{v}_{\ln 1 = 0} \frac{\partial u}{\partial r} \, ds = \int_{C_1} u \, ds$$

Inner:
$$\int_{C_\varepsilon} u \underbrace{\frac{\partial v}{\partial r}}_{-\frac{1}{\varepsilon}} - \underbrace{\underbrace{v}_{\ln \varepsilon} \underbrace{\frac{\partial u}{\partial r}}_{\text{something bounded}}}$$ $$ds$$

$$\leq \int |\ln \varepsilon| \cdot M \underbrace{ds}$$

length of C_ε is $2\pi\varepsilon$

$$\leq |\ln \varepsilon| M \cdot 2\pi\varepsilon \longrightarrow 0$$

as $\varepsilon \to 0$

Finally
$$\int_{C_\varepsilon} u \left(-\frac{1}{\varepsilon}\right) ds \qquad C_\varepsilon : (\varepsilon \cos\theta, \varepsilon \sin\theta)$$

$$= \int_0^{2\pi} u(\varepsilon \cos\theta, \varepsilon \sin\theta) \left(-\frac{1}{\varepsilon}\right) \underbrace{\varepsilon \, d\theta}_{ds}$$

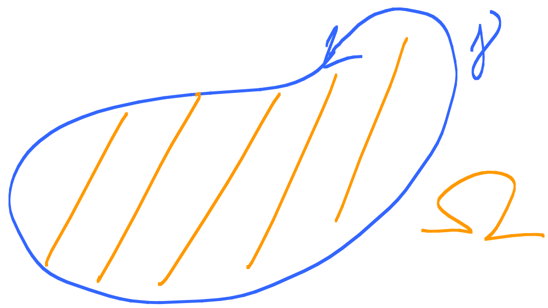
$$\longrightarrow u(0) \cdot 2\pi$$

Yes!

<u>Exciting hooks for Isop Ineq:</u>

Green's Identity:
$$\int_\gamma -y\,dx + x\,dy = \iint_\Omega \underbrace{1}_{\frac{\partial Q}{\partial x}} - \underbrace{(-1)}_{\frac{\partial P}{\partial y}} \, dx\,dy$$

Aha! $\frac{1}{2}\int_\gamma -y\,dx + x\,dy = \text{Area}(\Omega)$



γ: $x(t),\ y(t) \quad -\pi \le t \le \pi$

$$\frac{1}{2}\int_\gamma -y\,dx + x\,dy = \frac{1}{2}\int_{-\pi}^{\pi} -y(t)\,\dot{x}(t) + x(t)\,\dot{y}(t)\,dt$$

inner product

<u>Fact</u>: Parseval's $\Longrightarrow$ $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\,dt = \sum_{-\infty}^{\infty} \hat{f}(n)\,\overline{\hat{g}(n)}$

Norm preserved $\Longrightarrow$ inner products preserved