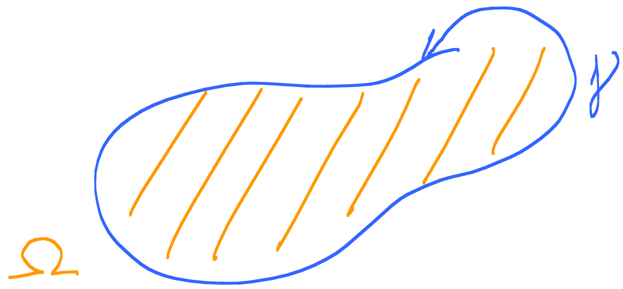
Lecture 24 The Isoparametric inequality via Fourier series!



$$A \le \frac{L^2}{4\pi}$$

A = Area of Ω, L = length of simple closed γ ← C^2-smooth

Parseval's : $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = \hat{f}(n)$

$$\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\, dt}_{L^2[-\pi,\pi] \text{ norm}} = \underbrace{\sum_{-\infty}^{\infty} |\hat{f}(n)|^2}_{\ell^2 \text{ norm}}$$

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt \approx \langle \{c_n\}, \{d_n\}_{-\infty}^{\infty} \rangle = \sum_{-\infty}^{\infty} c_n \overline{d_n}$$

Fact: Isometry $\Rightarrow$ $\langle\ ,\ \rangle$ preserved!

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\, dt = \sum_{-\infty}^{\infty} \hat{f}(n)\,\overline{\hat{g}(n)}}$$

Key fact #1

Fact: f is C^2, then $\widehat{f'}(n) = in\,\hat{f}(n)$ ← Key fact #2

How to remember : $f(t) = \sum_{-\infty}^{\infty} \hat{f}(n)\, e^{int}$

$$f'(t) \sim \sum_{-\infty}^{\infty} \underbrace{in\,\hat{f}(n)}_{\widehat{f'}(n)}\, e^{int}$$ ← wishful thinking

How to prove: $\hat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)\, e^{-int}\, dt$

$= \ldots$ integration by parts $\ldots$

$= \frac{1}{2\pi}\, in \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$ ✓

Green's Theorem fact: $\frac{1}{2}\int_\gamma \underset{\substack{\uparrow \\ P=-y}}{-y}\, dx + \underset{\substack{\uparrow \\ Q=x}}{x}\, dy = \text{Area}(\Omega)$

$= \frac{1}{2}\iint_\Omega \underbrace{\frac{\partial Q}{\partial x}}_{1} - \underbrace{\frac{\partial P}{\partial y}}_{-1}\, dx\,dy = \iint_\Omega 1\, dx\,dy$ ✓

Step 1: $\gamma : x(t), y(t)$ parametrized with respect to arc length t.

γ C^2-smooth means x, y C^2-smooth and periodic.

Step 2: Rescale so that γ has length 2π.

$$\gamma : \begin{cases} x(t) = \sum_{-\infty}^{\infty} \hat{x}(n)\, e^{int} \\ y(t) = \sum_{-\infty}^{\infty} \hat{y}(n)\, e^{int} \end{cases} \qquad -\pi \le t \le \pi$$

Step 3: t arc length. So $\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} = 1$

and so $\dot{x}^2 + \dot{y}^2 \equiv 1$ too!

Step 4: Do it! Want $A \le \pi$.

$$A = \frac{1}{2}\int_\gamma -y\,dx + x\,dy$$

$$\boxed{dx = \frac{dx}{dt}\,dt}$$

$$= \frac{1\cdot\pi}{2\pi}\int_{-\pi}^{\pi} -y(t)\,\dot{x}(t) + x(t)\,\dot{y}(t)\,dt$$

$$= \frac{1\cdot\pi}{2\pi}\left[\langle -y, \dot{x}\rangle + \langle x, \dot{y}\rangle\right]$$

$$\underset{\substack{\uparrow\\ \text{Parseval's}}}{=} \pi\sum_{-\infty}^{\infty}\left(-\hat{y}(n)\,\overline{[in\,\hat{x}(n)]} + \hat{x}(n)\,\overline{[in\,\hat{y}(n)]}\right)$$

$$= \pi\sum_{-\infty}^{\infty} in\left(\hat{y}(n)\,\overline{\hat{x}(n)} - \hat{x}(n)\,\overline{\hat{y}(n)}\right)$$

Next: $$\int_{-\pi}^{\pi}\underbrace{\dot{x}(t)^2 + \dot{y}(t)^2}_{1}\,dt = 2\pi$$

$$\underset{\substack{\uparrow\\ \text{Parseval's}}}{=} 2\pi\sum_{-\infty}^{\infty}\left|[in\,\hat{x}(n)]\right|^2 + \left|[in\,\hat{y}(n)]\right|^2$$

$$\pi \sum_{-\infty}^{\infty} n^2\left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2\right) = 1 \cdot \pi$$

Finally,

$$\pi - A = \text{Sum of squares!}$$

$$\pi \sum_{-\infty}^{\infty} \underbrace{n^2\left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2\right)}_{n^2 \text{ of these}} - \pi \sum_{-\infty}^{\infty} in\left(\hat{y}(n)\overline{\hat{x}(n)} - \hat{x}(n)\overline{\hat{y}(n)}\right)$$

move one over here

Write $\hat{x}(n) = \alpha_n + i\beta_n$

$\hat{y}(n) = a_n + i b_n$

Algebra: $$\pi - A = \pi \sum_{-\infty}^{\infty}\left[n^2\left(\alpha_n^2 + \beta_n^2 + a_n^2 + b_n^2\right) - 2n\left(\beta_n a_n - \alpha_n b_n\right)\right]$$

$$= \pi \sum_{\substack{-\infty \\ n \neq 0}}^{\infty}\left[(n\alpha_n + b_n)^2 + (n\beta_n - a_n)^2 + (n^2-1)(a_n^2 + b_n^2)\right]$$

Aha! = Sum of squares! So ≥ 0. ✓

What if $\tilde{\pi} - A = 0$?

Then $a_n = b_n = 0 \quad |n| \geq 2$

$\Longrightarrow \alpha_n, \beta_n = 0 \quad |n| \geq 2$ too.

For $n = \pm 1$, we have

$$\alpha_{\pm 1} = \mp b_{\pm 1} \qquad \beta_{\pm 1} = a_{\pm 1}$$

So
$$\begin{cases} x(t) = \hat{x}(0) + 2\alpha_1 \cos t - 2\beta_1 \sin t \\ y(t) = \hat{y}(0) + 2\beta_1 \cos t + 2\alpha_1 \sin t \end{cases}$$

Write $z(t) = x(t) + i y(t) = \underbrace{\left(\hat{x}(0) + i\hat{y}(0)\right)}_{\text{center}} + c\, e^{it}$

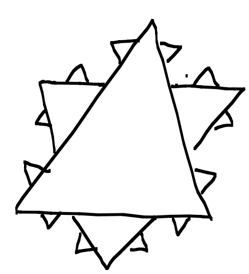
complex #
$c = Re^{i\theta}$

See this is a <u>circle</u>.

<u>Remarks:</u> Proof of Green's Thm in Spivak, Calculus on manifolds.

How bad can the Isop. ineq go?

Sierpinski's snowflake



Dym & Mckean. Fourier analysis.

$L \to \infty$
A bounded!