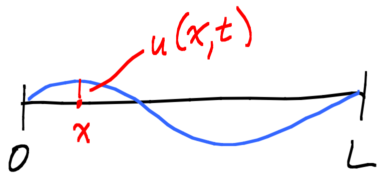
Lecture 26 D'Alembert's solution of the string problem



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

BC: $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ IC $\quad u(x,0) = f(x)$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Johann's $\rightsquigarrow$ $g(x) \equiv 0$ case:

Jacob's $\longrightarrow$ $u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct) \right]$

where f^* is the odd $2L$-periodic extension of f to $\mathbb{R}$.

IC ✓ $\quad u(x,0) = \frac{1}{2}\left[f^*(x) + f^*(x) \right] = f(x)$ ✓

BC $\quad u(0,t) = \frac{1}{2}\left[f^*(ct) + f^*(-ct) \right]$ ✓

f^* odd: $f^*(-ct) = -f^*(ct)$ $\quad (*)$

$$u(L,t) = \frac{1}{2}\left[\underbrace{f^*(L+ct) + f^*(L-ct)}_{=0 \text{ because of } (*) \text{ and } 2L\text{-periodic}} \right] \checkmark$$

What if g not the zero fcn? Let f^* and g^* be the odd $2L$-periodic extensions of f and g.

Let $G(x) = \int_0^x g(u)\,du.$ $\quad [G'(x) = g(x)].$

<u>Sol^n</u>: $u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct)\right]$

$$+ \frac{1}{2c}\left[G(x+ct) - G(x-ct)\right]$$

<u>D'Alembert's method</u>: $\underbrace{\xi}_{v} = x+ct$, $\underbrace{\eta}_{w} = x-ct$

$u =$ fcn of x, t.

$$\underset{t}{u_x} = u_v \cdot \underbrace{\frac{\partial v}{\partial x}}_{c} + u_w \cdot \underbrace{\frac{\partial w}{\partial x}}_{-c} = u_v \underset{-}{+} u_w$$

$$\underset{tt}{u_{xx}} = \left(u_{vv}\underbrace{\frac{\partial v}{\partial x}}_{c} + u_{wv}\underbrace{\frac{\partial w}{\partial x}}_{-c}\right) \pm \left(u_{vw}\underbrace{\frac{\partial v}{\partial x}}_{c} + u_{ww}\underbrace{\frac{\partial w}{\partial x}}_{-c}\right)$$

equal

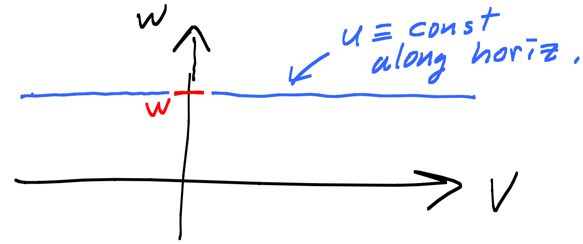
$$= \underline{u_{vv} + 2u_{vw} + u_{ww}}$$

$$u_{tt} = c^2\left[u_{vv} - 2u_{vw} + u_{ww}\right] \overset{\text{want}}{=} c^2\left[u_{vv} + 2u_{vw} + u_{ww}\right]$$

In v,w coords, the wave equ becomes

$$\boxed{u_{vw} = 0} \longleftarrow \text{after dividing by } 4c^2$$

Easiest PDE: $\dfrac{\partial u}{\partial v} \equiv 0$



So $u(v,w)$ is a fcn of w alone.

Sol^n: $u(v,w) = C(w)$ ← arbitrary fcn of w.

Our prob: $u_{vw} \equiv 0$

$$\frac{\partial}{\partial w}\left[\frac{\partial u}{\partial v}\right] \equiv 0$$

Conclude $\dfrac{\partial u}{\partial v} = C(v)$ ← fcn of v

Hmmm. Let $p(v)$ be an antiderivative of $C(v)$.

$$\left[\, p(v) = \int_0^v C(s)\,ds \,.\right]$$

Then $\dfrac{\partial u}{\partial v} = p'(v)$

$$\frac{\partial}{\partial v}\left[u - p\right] \equiv 0$$

Aha! So $u - p(v) = K(w)$ ← fcn of w alone.

Done: $u(v,w) = \underset{\varphi}{p}(v) + \underset{\psi}{K}(w)$

Get $\quad u(x,y) = \varphi(x+ct) + \psi(x-ct)$

PDE ✓ easy.

BC ✓ yes if φ, ψ are odd and $2L$-periodic.

IC? $\quad \dfrac{\partial u}{\partial t}(x,y) = c\varphi'(x+ct) - c\psi'(x-ct)$

want

Need: $\begin{cases} u(x,0) = \varphi(x) + \psi(x) = f(x) & (A) \\ \dfrac{\partial u}{\partial t}(x,0) = c\varphi'(x) - c\psi'(x) = g(x) & (B) \end{cases}$

Integrate (B) $\int_0^x$: $\quad c\varphi(x) - c\psi(x) = \int_0^x g(s)\,ds = G(x)$

Assume $\varphi(0)$ $\psi(0)$ $\qquad$ define

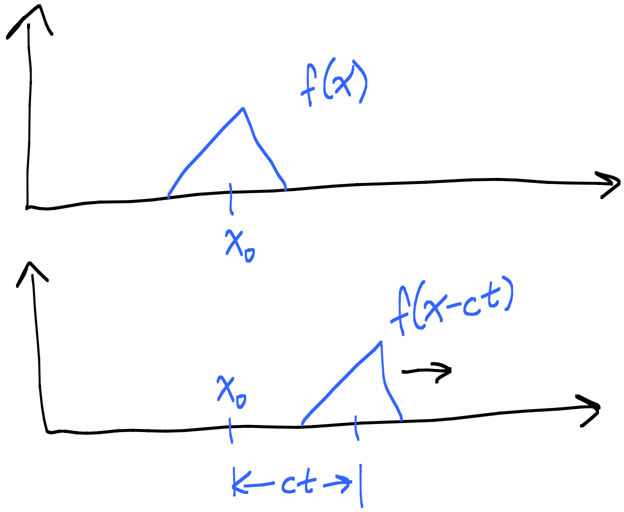
(B')

(A) $\quad \varphi(x) + \psi(x) = f(x)$

$\frac{1}{c}$(B') $\quad \varphi(x) - \psi(x) = \frac{1}{c} G(x)$

$+$: $\quad \varphi(x) = \frac{1}{2}\left[f(x) + \frac{1}{c} G(x)\right]$

$-$: $\quad \psi(x) = \frac{1}{2}\left[f(x) - \frac{1}{c} G(x)\right]$



$f(x-ct)$ waveforms moving to right.
$f(x+ct)$ move to left.

$L^2[-\pi,\pi] \approx \ell^2$ via Parseval's.

$$\underbrace{\frac{1}{2\pi}\left(\|f+g\|^2 - \|f-g\|^2\right)}_{\frac{1}{2\pi}\int_{-\pi}^{\pi}(f+g)\overline{(f+g)}\,dt} = \sum_{-\infty}^{\infty} + \sum_{-\infty}^{\infty}$$

$$f\cdot\bar{f} + \underbrace{g\bar{f} + \bar{g}f}_{2\mathrm{Re}\, f\bar{g}} + g\bar{g}$$

$$|f|^2 \qquad\qquad |g|^2$$

Get

$$\mathrm{Re}\left[\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx - \sum_{-\infty}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}}_{\text{complex number } re^{i\theta}}\right] = 0$$

Aha! Replace f by $e^{-i\theta}f$.

Or replace f by $-if$ to see Im part $=0$ too.