Lecture 27 Fundamental Theorem of Algebra

HWK 7 due Mon.

$$u(x,t) = \tfrac{1}{2}\left(f^*(x-ct) + f^*(x+ct)\right)$$

$$\text{BC}\begin{cases} u(x,0) = \tfrac{1}{2}\Big(\underbrace{f^*(-ct) + f^*(ct)}_{=0 \text{ because } f^* \text{ odd}} \Big) \\ u(L,0) = \tfrac{1}{2}\Big(\underbrace{f^*(L-ct)}_{= f^*(-L-ct)} + f^*(L+ct) \Big) \end{cases}$$

$\leftarrow$ zero because the odd $2L$-periodic ext is odd

Fund. Thm Alg: Given a polynomial $P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$ of degree $N \geq 1$, $P(z)$ has a root in $\mathbb{C}$.

Analytic functions:

EX: $f(z) = e^z = e^{x+iy} = e^x e^{iy} = \underbrace{e^x \cos y}_{u(x,y)} + i\,\underbrace{e^x \sin y}_{v(x,y)}$

EX: $f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2 - y^2)}_{u(x,y)} + i\,\underbrace{(2xy)}_{v(x,y)}$

Def^n: Given an open set $\Omega \subset \mathbb{C}$, f is analytic on Ω if it is complex diff'ble, meaning

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \quad \text{exists at each } a \in \Omega.$$

EX: $f(z) = z^2$:
$$\frac{f(z) - f(a)}{z - a} = \frac{z^2 - a^2}{z - a} = \frac{(z-a)(z+a)}{z - a}$$
$$= z + a \longrightarrow 2a \quad \text{as } z \to a.$$

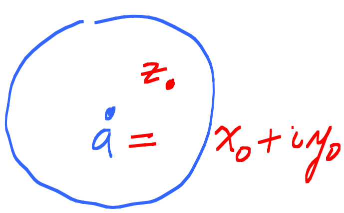
Just like real calculus! $f'(z) = 2z$.

Fact: All basic rules of diff'tion $\mathbb{R} \to \mathbb{R}$ go over to $\mathbb{C} \to \mathbb{C}$. [e.g.: $(f/g)' = \frac{f'g - g'f}{g^2}$ where g non-vanishing, etc.]

Fact: Polynomials are analytic on $\mathbb{C}$. Complex rational fcns are analytic on $\mathbb{C}$ minus points where denom $= 0$.

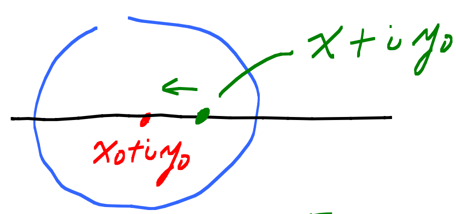
Cauchy-Riemann Equations: $f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic, Then

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{CR-Eqns.}$$



$$DQ = \frac{f(z) - f(a)}{z - a} \to f'(a)$$

z slides in to a along any direction

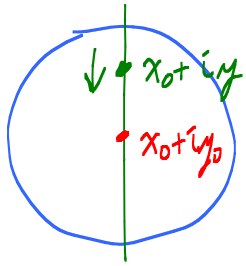


$$DQ = \frac{[u(x,y_0) + i\,v(x,y_0)] - [u(x_0,y_0) + i\,v(x_0,y_0)]}{(x+iy_0) - (x_0+iy_0)}$$

$$= \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i\,(\text{DQ for } v)$$

$$\to \boxed{u_x(x_0,y_0) + i\,v_x(x_0,y_0) = f'(a)}$$

red box #1



$$DQ = \frac{\big(u(x_0,y)+iv(x_0,y)\big) - \big(u(x_0,y_0)+iv(x_0,y_0)\big)}{\underbrace{(x_0+iy)-(x_0+iy_0)}_{i(y-y_0)}}$$

$$= \frac{1}{i}\Big[(DQ \text{ for } u) + i\,(DQ \text{ for } v)\Big]$$

$$\longrightarrow \boxed{v_y(x_0,y_0) - i\,u_y(x_0,y_0) = f'(a)}$$

red box #2.

Equate red boxes. Get C-R Equs.

Remark: u, v are C^1-smooth and satisfy C-R Equs.

Then $f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic.

Big fact: Real and imaginary parts of analytic fcns are harmonic!

Proof for polys and rational fcns (which have C^∞-smooth real and imag parts).

$\underbrace{R(x+iy)}_{\text{rational or poly}} = u(x,y) + i\,v(x,y)$ analytic ✓

So C-R Equs hold: $\begin{cases} u_x = v_y & \text{(A)} \\ u_y = -v_x & \text{(B)} \end{cases}$

$\frac{\partial}{\partial x}(A)$: $\quad u_{xx} = v_{xy}$

$\frac{\partial}{\partial y}(B)$: $\quad u_{yy} = -v_{yx}$

$$\boxed{u, v\ C^2\text{-smooth} \implies v_{xy} = v_{yx}}$$

$(+)$ $\quad u_{xx} + u_{yy} = 0$ ✓ $\quad u$ harm.

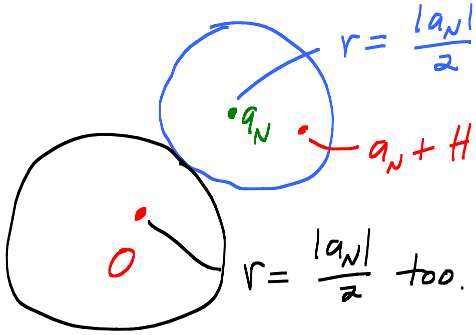
Similarly, so is v. $\left[\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B) = \Delta v \equiv 0.\right]$

<u>Lemma:</u> $\lim_{|z|\to\infty} |P(z)| = \infty$ if P poly of deg $N \geq 1$.

<u>Pf:</u> $P(z) = a_N z^N + \cdots + a_0, \quad a_N \neq 0$

$$= z^N \Big[a_N + \underbrace{\frac{a_{N-1}}{z} + \cdots + \frac{a_0}{z^N}}_{H(z)} \Big]$$

Note $H(z) \to 0$ as $|z| \to \infty$.



$a_N + H(z)$ when $|H(z)| < r = \frac{|a_N|}{2}$

when $|z| > R$.

$$\underbrace{\frac{|a_N|}{2}}_{A} \leq \left|\frac{P(z)}{z^N}\right| = \left| a_N + H(z) \right| \leq \underbrace{\frac{3|a_N|}{2}}_{B}$$

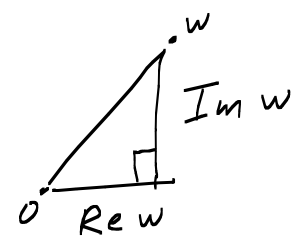
Conclude $\underbrace{A|z|^N \le |P(z)|}_{(*)} \le B|z|^N$ when $|z| > R$.

This one shows $|P(z)| \to \infty$ when $|z| \to \infty$.

Pf of F. Thm Alg: Suppose $P(z)$, deg $N \ge 1$, has no zeroes. Then $\frac{1}{P(z)}$ is analytic on $\mathbb{C}$.

(*) Shows that $\left|\frac{1}{P(z)}\right| \le \frac{1}{A|z|^N} \to 0$ as $|z| \to \infty$.

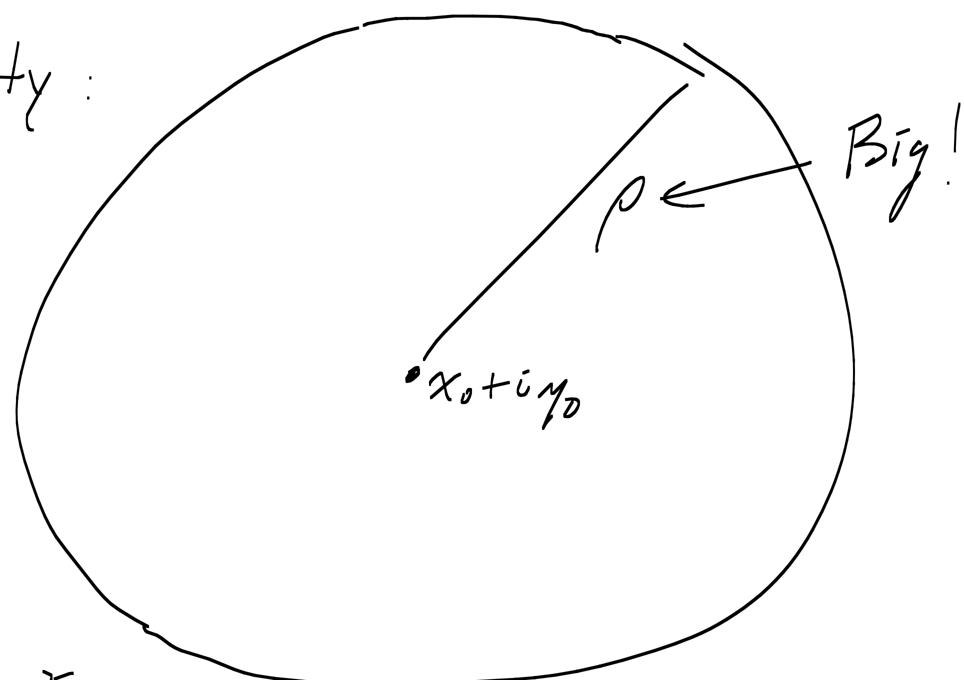
But $\frac{1}{P(x+iy)} = \underbrace{u(x,y) + i\,v(x,y)}_{\text{harmonic fcns that} \to 0 \text{ as } \sqrt{x^2+y^2} \to \infty}$.



$|\mathrm{Re}\, w| \le |w|$

$|\mathrm{Im}\, w| \le |w|$

Aha! Ave property:



$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(x_0 + \rho\cos\theta,\ y_0 + \rho\sin\theta)}_{|\cdot| \le \varepsilon}\, d\theta$$

See $|u(x_0, y_0)|$ and $|v(x_0, y_0)| \le \varepsilon$.

ε arbitrary! Conclude $u \equiv 0$, $v \equiv 0$. $\lightning$.

[P never zero. $\frac{1}{P}$ is never zero too!]

EX: x^2+1

Tossing in $i = \sqrt{-1}$ makes this and all polys have a full set of roots. $a + bi$