Lecture 29   Real Fourier Transform   Hwk 7 due Mon

Couple last remarks about Laplace transform:

$$\mathcal{L}[f'] = sF(s) - f(0)$$

$F = \mathcal{L}[f]$

Why: $$\mathcal{L}[f'] = \int_0^\infty f'(t)\, \underset{u}{e^{-st}}\, dt = \lim_{B\to\infty}\left[\underbrace{\left(e^{-st}f(t)\right)\Big|_0^B}_{0 - f(0)} - \underbrace{\int_0^B f(t)\left(-se^{-st}\right)dt}_{sF(s)} \right]$$

$dv = f'(t)\,dt$

$v = f(t)$

when $s > K$: $|f(t)| \le Me^{Kt}$

Formulas:

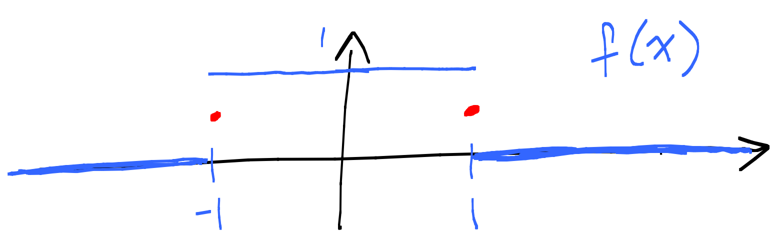
$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

$$\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2} \qquad \mathcal{L}[e^{ibt}] = \frac{1}{s-ib}$$

$$\mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}$$

$$\left.\begin{aligned} \mathcal{L}[e^{-at}\cos bt] &= \frac{s+a}{(s+a)^2+b^2} \\ \mathcal{L}[e^{-at}\sin bt] &= \frac{b}{(s+a)^2+b^2} \end{aligned}\right\} \quad \mathcal{L}[e^{-at}f(t)] = F(s+a)$$

Fourier Transform example:



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{even}}\ \underbrace{\cos wt}_{\text{even}}}_{\text{even}}\, dt = 2\,\frac{1}{\pi}\int_0^{\infty} f(t)\cos wt\, dt$$

$$= \frac{2}{\pi}\int_0^1 1\cdot \cos wt\, dt = \frac{2}{\pi}\left[\frac{1}{w}\sin wt\right]_0^1$$

$$= \boxed{\frac{2}{\pi}\ \frac{\sin w}{w}} \quad A(w)$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{even}}\ \underbrace{\sin wt}_{\text{odd}}}_{\text{odd}}\, dt = 0$$

Now

$$f(x) \text{ "=" } \int_0^{\infty} \underbrace{\frac{2}{\pi}\ \frac{\sin w}{w}}_{A(w)} \cos wx\, dw + \int \underbrace{B(w)}_{\equiv 0} \cdots$$

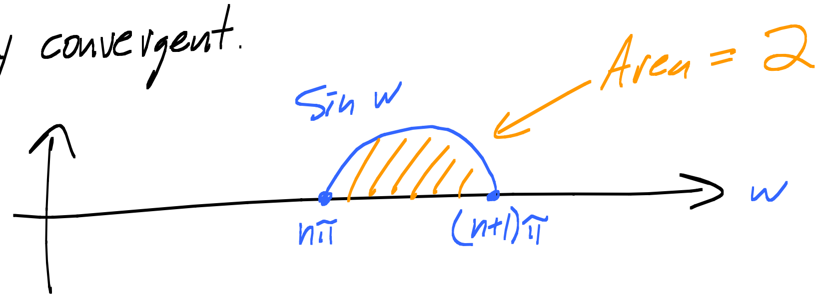
$$= \begin{cases} 0 & x < -1 \\ 1/2 & x = -1 \\ 1 & -1 < x < 1 \\ 1/2 & x = 1 \\ 0 & x > 1 \end{cases}$$

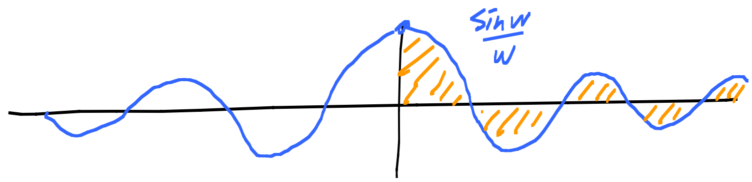
<u>Aha!</u> Let $x = 0$: Get $1 = \frac{2}{\pi}\int_0^{\infty} \frac{\sin w}{w}\, dw$

$$\boxed{\int_0^{\infty} \frac{\sin w}{w}\, dw = \frac{\pi}{2}}$$

Famous integral! Not absolutely convergent.

Conditionally convergent.





Alternating series test shows that $\lim_{B \to \infty} \int_0^B \frac{\sin w}{w}\, dw$ exists.

But
$$\frac{|\sin w|}{(n+1)\pi} \le \frac{|\sin w|}{w} \le \frac{|\sin w|}{n\pi}$$

Integrate $n\pi : (n+1)\pi$. Get

$$\frac{2}{(n+1)\pi} \le \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin w}{w} \right| dw \le \frac{2}{n\pi}$$

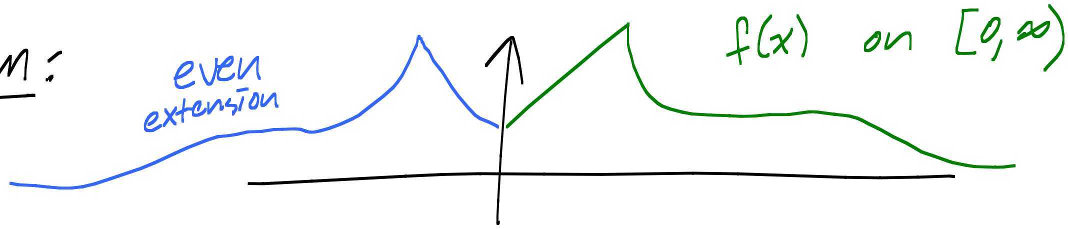
Sum: $0 : N$. Conclude: $\int_0^{N\pi} \left| \frac{\sin w}{w} \right| dw$

$\longrightarrow \infty$ exactly like $\frac{2}{\pi} \sum_{n=1}^{N} \frac{1}{n} \longrightarrow \infty$ Harmonic Series!

Fact: f even $\Longrightarrow B(w) \equiv 0$

f odd $\implies A(w) \equiv 0$

Hmmm:



$$f(x) = \int_0^\infty A(w) \cos wx \, dw \quad \text{where} \quad A(w) = \frac{2}{\pi}\int_0^\infty f(t) \cos wt \, dt$$

Aha!: Define "Fourier Cosine Transform" to be

$$\tilde{\mathcal{F}}_c[f](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos wt \, dt$$

Wow! $\quad (\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c)[f] = f$

Note: $\tilde{\mathcal{F}}_c$ makes sense for f defined on $[0,\infty)$ and maps into same family of fcns.

Def^n: Fourier Sine Transform:

$$\tilde{\mathcal{F}}_s[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \underline{\underline{\sin}}\, wt \, dt$$