Lecture 30 Fourier Cosine and Sine Transforms

$$\tilde{\mathcal{F}}_c[f](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx \leftarrow$$ using even extension of f

$$\tilde{\mathcal{F}}_s[f](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx \leftarrow$$ using odd ext.

Key properties:

1) $$\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c = \text{identity}$$

$$\tilde{\mathcal{F}}_s \circ \tilde{\mathcal{F}}_s = \text{identity}$$

2) $$\tilde{\mathcal{F}}_c[f'] = w\,\tilde{\mathcal{F}}_s[f] - \sqrt{\frac{2}{\pi}}\, f(0) \leftarrow$$ need $f(x) \to 0$ as $x \to \infty$.

$$\tilde{\mathcal{F}}_s[f'] = -w\,\tilde{\mathcal{F}}_c[f]$$

3) $$\tilde{\mathcal{F}}_c[f''] = -w^2\,\tilde{\mathcal{F}}_c[f] - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$\tilde{\mathcal{F}}_s[f''] = -w^2\,\tilde{\mathcal{F}}_s[f] + \sqrt{\frac{2}{\pi}}\, f(0)\, w$$

Why: $$\tilde{\mathcal{F}}_c[f'](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \underbrace{\cos wx}_{u} \, dx$$

$dv = f'(x)\,dx$

$v = f(x)$

$$= \sqrt{\frac{2}{\pi}} \lim_{B\to\infty} \left[(\cos wx) f(x) \Big|_0^B - \int_0^B f(x) \left[-w \sin wx\right] dx \right]$$

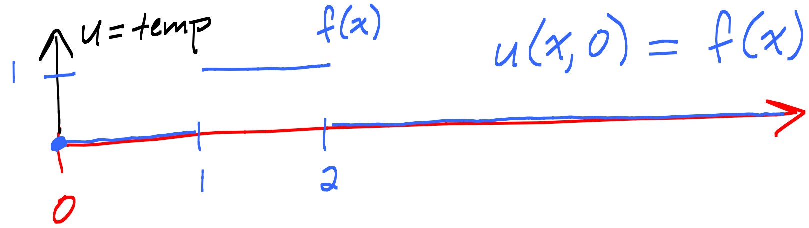
need $f(x) \to 0$ as $x \to \infty$.

$$= -\sqrt{\frac{2}{\pi}}\, f(0) + w\,\tilde{\mathcal{F}}_s[f] \checkmark$$

Similarly for part 2 b. (Easier because $\sin 0 = 0$.)

(3) follows from 2. ✓

Application: Hot wire



$u(x,0) = f(x)$

Heat Eqn: $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

Initial Cond: $u(x,0) = f(x)$

Boundary Cond: $u(0,t) = 0$ ← homogeneous B.C.

Big idea: Take $\tilde{\mathcal{F}}_s$ in the space variable.

$$\text{Write} \quad \hat{u}(w,t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty u(x,t) \sin wx \, dx$$

PDE?

$$\frac{\partial \hat{u}}{\partial t} = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial u}{\partial t}(x,t)}_{\frac{\partial^2 u}{\partial x^2}} \sin wx \, dx$$

$$= \tilde{\mathcal{F}}_s \left[\frac{\partial^2 u}{\partial x^2} \right]$$

$$= -w^2 \underbrace{\tilde{\mathcal{F}}_s [u]}_{\hat{u}} + \sqrt{\tfrac{2}{\pi}} \underbrace{u(0,t)}_{=0} w$$

New PDE: $\boxed{\dfrac{\partial \hat{u}}{\partial t} = -w^2 \hat{u}}$ ← exp decay ODE in t variable

Easy! $\hat{u}(w,t) = \underbrace{C(w)}\, e^{-w^2 t}$

this const. depends on w.

Remark: $\frac{\partial}{\partial t}\left[\frac{\hat{u}}{\underbrace{e^{-w^2 t}}}\right] \equiv 0$

this is a fcn of w alone. Call it $C(w)$.

Next: Determine $C(w)$ by plugging in $t=0$.

$$\hat{u}(w,0) = C(w)e^{-w^2\cdot 0} = C(w)$$

So $C(w) = \hat{u}(w,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty \underbrace{u(x,0)}_{f(x)\ (\text{I.C.})} \sin wx\, dx$

$$= \sqrt{\frac{2}{\pi}}\int_1^2 1\cdot \sin wx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{1}{w}\cos wx\right]_{x=1}^{2}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\cos w - \cos 2w}{w}\right) = C(w)$$

Last step: Undo $\tilde{\mathcal{F}}_s$ by applying $\tilde{\mathcal{F}}_s$ again!

$$u(x,t) = \tilde{\mathcal{F}}_s \underbrace{\tilde{\mathcal{F}}_s u}_{\hat{u}(w,t) = C(w)e^{-w^2 t}}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right) e^{-w^2 t} \sin wx \, dw$$

Note: $\frac{\cos w - \cos 2w}{w} = \frac{\left(1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \cdots\right) - \left(1 - \frac{4w^2}{2!} + \frac{16w^4}{4!} - \cdots\right)}{w}$

$= \frac{3}{2!} w +$ regular power series. $\leftarrow$ nice as $w \to 0$.

Remark: Insulated end at $x=0$: No heat flow past $=0$

Temp gradient $\Big|_{x=0} \equiv 0$

$\frac{\partial u}{\partial x}(x,0) \equiv 0 \leftarrow$ use $\tilde{\mathcal{F}}_c$ instead of $\tilde{\mathcal{F}}_s$.

EX: Calculate $\tilde{\mathcal{F}}_c[e^{-2x}]$

Aha! Think $s=2$ in Laplace Transf!

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cos xw \, dx$$

$$= \sqrt{\frac{2}{\pi}} \mathcal{L}[\cos wx] \Big|_{s=2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + w^2} \Big|_{s=2} = \underline{\underline{\sqrt{\frac{2}{\pi}} \cdot \frac{2}{w^2+4}}}$$

Hmmm. $e^{-2x} = \tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c[e^{-2x}] = \frac{2}{\pi} \int_0^\infty \frac{2}{w^2+4} \cos wx \, dw$!