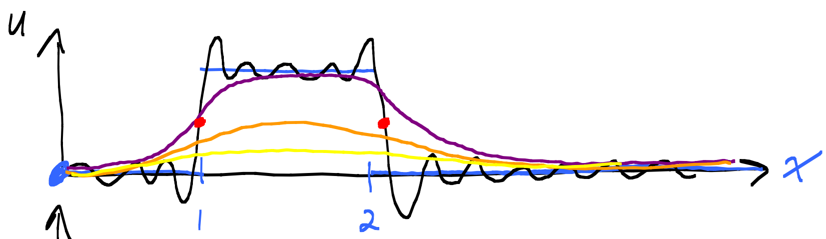
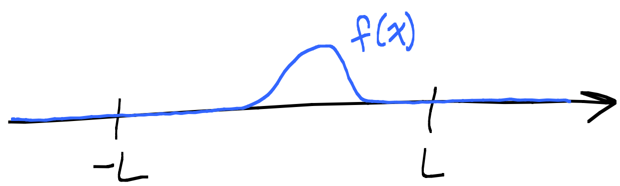
Lecture 31 The complex Fourier Transform



BC $u(0,t)=0 \leftarrow$ use $\mathcal{F}_s$ (refrigerated end)

If BC: $\frac{\partial u}{\partial x}(0,t)=0 \leftarrow$ use $\mathcal{F}_c$ (insulated end)

Complex F.T.:



Fourier series: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-int}\, dt$$

Write out. Let $L \to \infty$. Recognize Riemann integral.

Guess that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{ixs}\, ds$$

$\leftarrow$ Fourier inversion formula $= \mathcal{F}^{-1}[F]$

where $$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-its}\, dt$$

Fourier transform of f

Write $F(s) = \mathcal{F}[f] = \hat{f}$

$$f(x) = \mathcal{F}^{-1}[\hat{f}] = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx}\, ds$$

Parseval's Identity : $\|f\|^2 = c\|\hat{f}\|^2$

$$\int_{-\infty}^{\infty} |f|^2 dx = c\int_{-\infty}^{\infty} |\hat{f}|^2 ds$$

c depends on where 2π gets put.

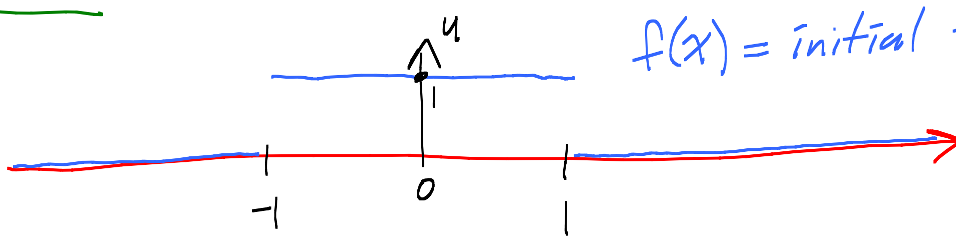
Notation : $\hat{f} = \mathcal{F}[f]$. $\check{g} = \mathcal{F}^{-1}[g]$

Big fact : $\check{\hat{f}} = f$

Big properties : $\mathcal{F}[f'] = is\hat{f}(s)$

$$\mathcal{F}[f''] = is\cdot \underbrace{\mathcal{F}[f']}_{is\,\mathcal{F}[f]} = -s^2 \hat{f}(s)$$

Heat problem : Hot infinite wire.



$f(x) =$ initial temp

PDE : $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ Initial cond : $u(x,0) = f(x)$. No BC.

Big sol^n with $\mathcal{F}$ in space variable x.

Write $\hat{u}(s,t) = \mathcal{F}[u] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\, e^{-ixs}\, dx$

Then (hopefully) $\frac{\partial \hat{u}}{\partial t} = \int_{-\infty}^{\infty} \underbrace{\frac{\partial u}{\partial t}(x,t)}_{\frac{\partial^2 u}{\partial x^2}(x,t)} e^{-ixs}\,dx$

$$\frac{\partial \hat{u}}{\partial t} = -s^2 \hat{u}(s,t)$$ ← exponential decay in t var.

So $\hat{u}(s,t) = C(s) e^{-s^2 t}$ ← different C for diff s perhaps.

Get C(s) from I.C. :

$$\hat{u}(s,0) = C(s) \cdot e^0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{f(x)} e^{-ixs}\,dx$$

$$= \frac{1}{2\pi} \int_{-1}^{1} 1 \cdot e^{-ixs}\,dx$$

$$= \frac{1}{2\pi} \left[\frac{1}{-is} e^{-ixs} \right]_{x=-1}^{1}$$

$$= \frac{1}{2\pi} \frac{e^{-is} - e^{-i(-s)}}{-is} = 2 \cdot \frac{e^{is} - e^{-is}}{2i} \cdot \frac{1}{s}$$

$$= \frac{1}{2\pi} \frac{2 \sin s}{s}$$

So $\hat{u}(s,t) = \frac{1}{2\pi} \frac{2 \sin s}{s} e^{-s^2 t}$

Last, undo $\wedge$ with $\vee$

$$u(x,t) = \underset{\text{using } s \text{ var}}{\tilde{\mathcal{F}}^{-1}}\left[\hat{u}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\,\frac{\sin s}{s}\, e^{-s^2 t}\, e^{ixs}\, ds$$

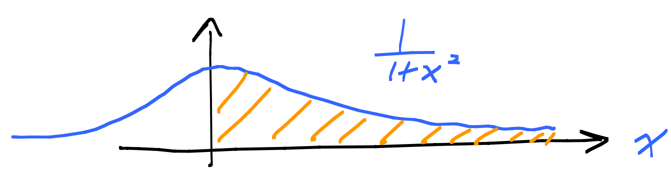
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin s}{s}}_{\text{even}}\ \underbrace{e^{-s^2 t}}_{\text{even in } s}\left(\underbrace{\cos xs}_{\text{even}} + i\,\underbrace{\sin xs}_{\text{odd!}}\right) ds$$

$$\int_{-\infty}^{\infty} = 0$$

(even)(even)(odd) = odd

Ans: $$\boxed{u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s}\cdot e^{-s^2 t}\cdot \cos xs \; ds}$$

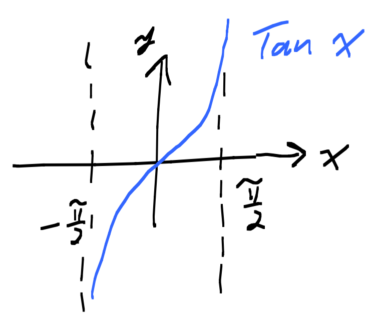
<u>Review of improper integrals:</u>

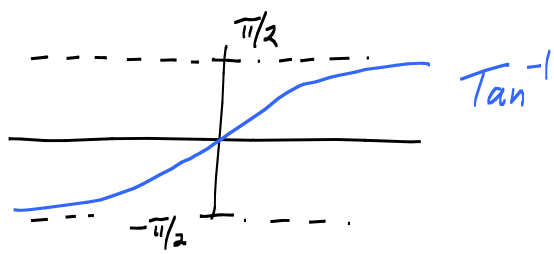


$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{B\to\infty}\int_0^B \frac{1}{1+x^2}\,dx \leftarrow$$

= ∞ or finite because fcn > 0

$$= \lim_{B\to\infty}\left[\operatorname{Tan}^{-1} x\right]_0^B = \frac{\pi}{2} - 0$$





<u>EX</u>: $\dfrac{\sin x}{x}$ more delicate. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is

<u>conditionally</u> convergent, but not absolutely conv.

<u>EX</u>: $\dfrac{\sin x}{1+x^2}$ is <u>abso</u>lutely conv: $\displaystyle\int_0^\infty |f|\,dx < \infty.$

<u>Fact</u>: Absolutely convergent integrals are convergent.

$$\int_0^\infty \frac{\sin x}{1+x^2}\,dx = \lim_{B\to\infty}\int_0^B \frac{\sin x}{1+x^2}\,dx \quad \text{makes sense.}$$