Lecture 32 Bump fcns

$\mathcal{M}(\mathbb{R})$ = "fcns on $\mathbb{R}$ of moderate decrease at infinity"

$$= \left\{ f : |f(x)| \le \frac{M}{1+x^2} \text{ for all } x, \; M > 0. \right\}$$

Since $\int_0^\infty \frac{1}{1+x^2} = \frac{\pi}{2} < \infty$, fcns in $\mathcal{M}$ are absolutely integrable.

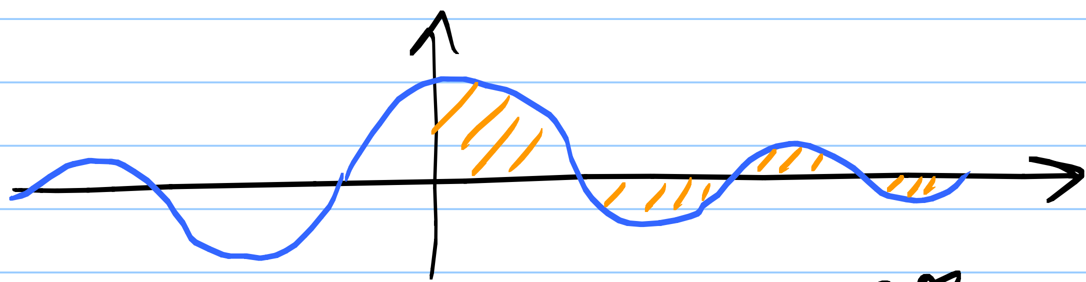
So
$$\int_{-\infty}^{\infty} f\,dx = \int_{-\infty}^{a} f\,dx + \int_{a}^{\infty} f\,dx$$

$$= \underbrace{\lim_{A \to -\infty} \int_A^a f\,dx}_{} + \underbrace{\lim_{B \to \infty} \int_a^B f\,dx}_{}$$

both limits exist if $f \in \mathcal{M}$.

And $\displaystyle\int_{-\infty}^{\infty} f \, dx = \lim_{N \to \infty} \int_{-N}^{N} f \, dx.$

<u>Note:</u> $\dfrac{\sin x}{x}$ is not in $\mathcal{M}$.

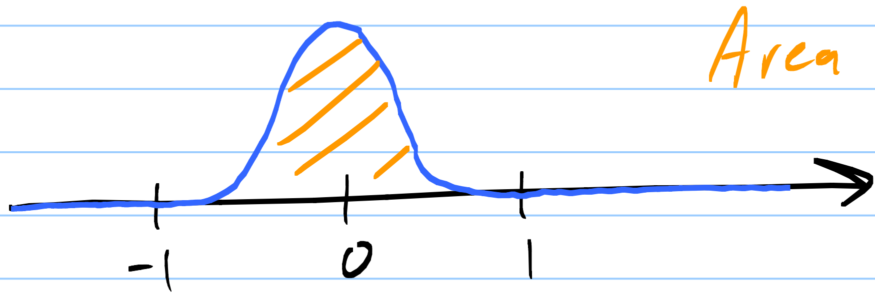


Conditionally convergent $\int_0^\infty$, not absolutely integrable.

<u>Integration facts for fcns in $\mathcal{M}$:</u>

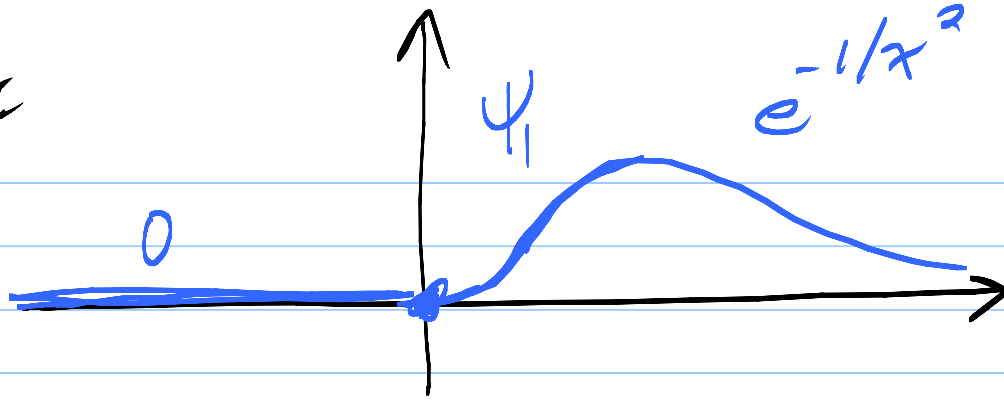
$$\int_{-\infty}^{\infty} f(x+h) \, dx = \int_{-\infty}^{\infty} f(x) \, dx$$

<u>Bump fcn in $C_0^\infty(\mathbb{R})$:</u>



Area = 1

Step 1:



Clear that ψ_1 is continuous at $x=0$

$$\psi_1' = e^{-1/x^2}\left[\frac{2}{x^3}\right]$$

$$= 2e^{-u} \cdot u^{3/2} \text{ where } u = \frac{1}{x^2} \to \infty$$

as $x \to 0$

e^{-u} dies away so fast, we see $\psi_1' \to 0$ as $x \to 0$.

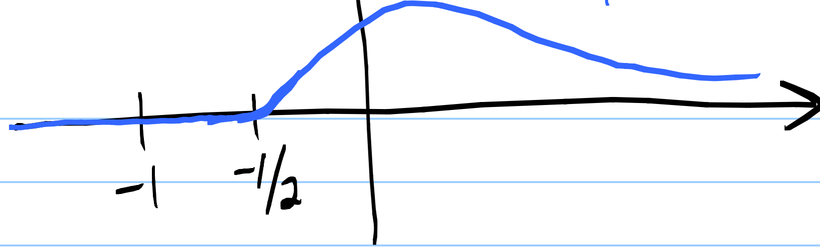
etc. See all derivatives of ψ_1 exist at 0.

So ψ_1 is C^∞ smooth on $\mathbb{R}$.

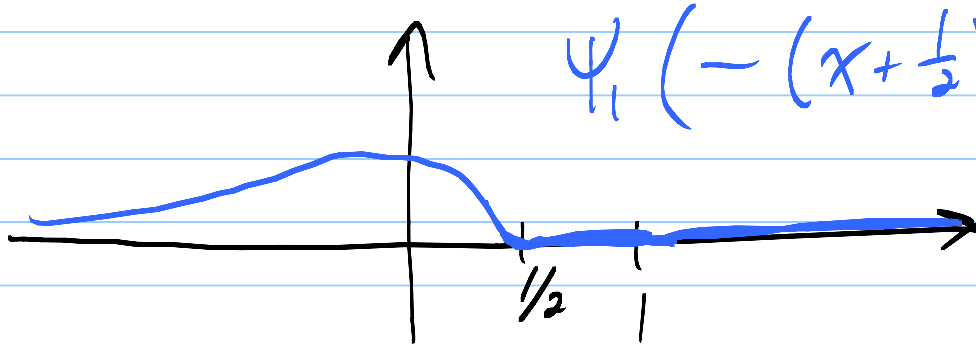
Fact: ψ_1 is not equal to its power series near $x=0$. [power series $\equiv 0$, but $\psi_1 \not\equiv 0$ near 0.]

Step 2: To get a "bump"

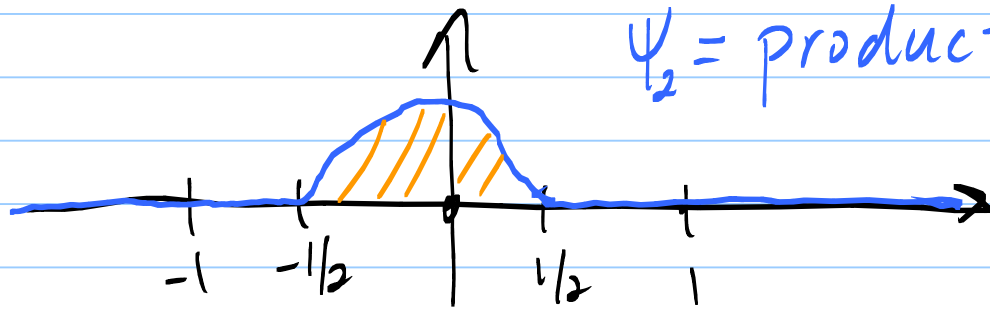
$\psi_1(x+\frac{1}{2})$



$\psi_1\left(-\left(x+\frac{1}{2}\right)\right)$



ψ_2 = product of above



Let $c = \int_{-\infty}^{\infty} \psi_2 \, dx = \int_{-1/2}^{1/2} \psi_2 \, dx > 0$

$\varphi = \frac{1}{c} \psi_2.$

Properties: $\varphi \in C_0^\infty(\mathbb{R})$

1) $\operatorname{supp} \varphi = \{x : \varphi(x) \neq 0\} \subset [-\frac{1}{2}, \frac{1}{2}]$

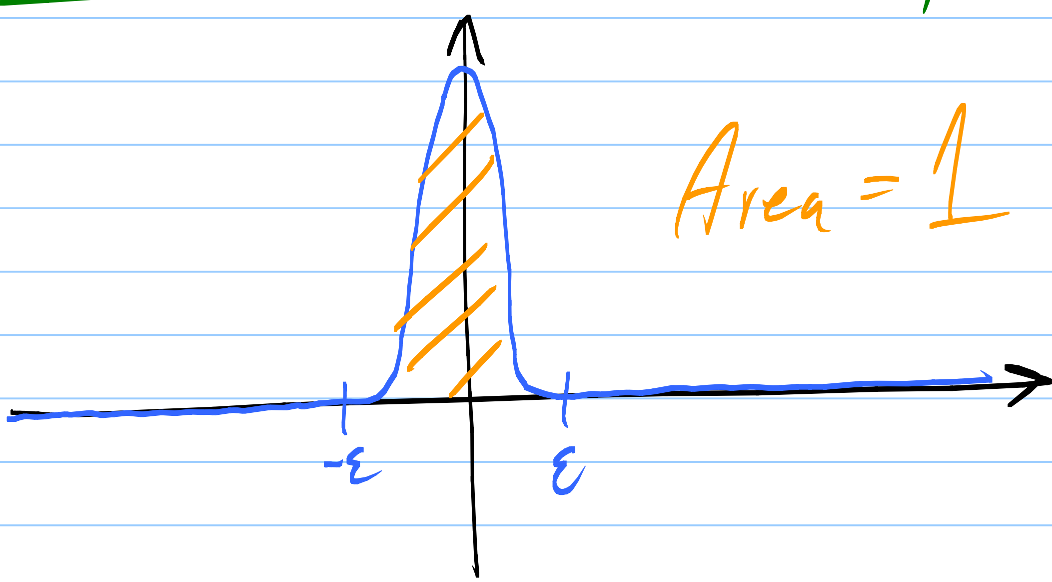
"φ has compact support"

2) Symmetric $\varphi(-x) = \varphi(x)$

3) $\displaystyle\int_{-\infty}^{\infty} \varphi \, dx = 1$

4) $\varphi(x) \geq 0$

Approximation to the identity:



$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$$

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(x)\, dx = \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(x)\, dx$$

$$= \int_{-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon}\, \varphi\left(\frac{x}{\varepsilon}\right) dx = \int_{-1}^{1} \varphi(u)\, du = 1$$

$$u = \frac{x}{\varepsilon} \qquad du = \frac{1}{\varepsilon}\, dx$$

When $x = -\varepsilon$: $u = \frac{-\varepsilon}{\varepsilon} = -1$

$x = \varepsilon$: $u = \frac{\varepsilon}{\varepsilon} = 1$

"Approx to identity": $\varphi_\varepsilon * f \approx f$

Convolution: $f * g(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$

Well defined when $f, g \in \mathcal{M}$.

Think about $(\varphi_\varepsilon * f)(x)$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(\underbrace{x-t}_{\tilde{\tau}})\, f(t)\, dt$$

$$d\tilde{\tau} = -dt$$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(\tilde{\tau})\, f(x-\tilde{\tau})\, d\tilde{\tau}$$

"Good kernel trick"

$$f(x) = f(x)\cdot 1 = f(x)\int_{-\infty}^{\infty} \varphi_\varepsilon(\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(\tau)\, f(x)\, d\tau$$

$$(\varphi_\varepsilon * f)(x) - f(x) =$$

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(\tau)\left[f(x-\tau) - f(x)\right] d\tau$$

$$= \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(\tau)\left[\underbrace{f(x-\tau) - f(x)}_{|\sim\sim|<E \text{ if } |\tau|<\delta.}\right] d\tau$$

Now $\left|(\varphi_\varepsilon * f) - f\right| \le \left|\int_{-\varepsilon}^{\varepsilon}\right| \le \underbrace{\int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(\tau)\, E\, d\tau}_{E\cdot \underbrace{\int_{-\varepsilon}^{\varepsilon}\varphi_\varepsilon(\tau)\,d\tau}_{1}}$

if $\varepsilon < \delta$.