Lecture 34 Fourier transform on $\mathcal{S}$: $\widehat{e^{-x^2}}$

Seeley: $\hat{f}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$

Fourier Integral: $f(x) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx}\, ds$

Stein: $\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi sx}\, dx$

$\widehat{e^{-\pi x^2}} = e^{-\pi s^2}$, $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$

Improper integral facts: Try exercises 4-1, 4-2 on p. 59.

Suppose $f \in \mathcal{M}(\mathbb{R})$. $\int_A^{\infty} f\, dx = \lim_{B\to\infty} \int_A^B f\, dx$

So $\underbrace{\int_A^{\infty} f\, dx}_{I} = \underbrace{\int_A^B f\, dx}_{\to I \text{ as } B\to\infty} + \underbrace{\int_B^{\infty} f\, dx}_{\text{Conclude this} \to 0 \text{ as } B\to\infty}$

Basic properties: $\int_{-\infty}^{\infty}$ on $\mathcal{M}(\mathbb{R})$

1) $f \mapsto \int_{-\infty}^{\infty} f\, dx$ is a linear operator

2) $$\int_{-\infty}^{\infty} f(x+h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

3) $$\int_{-\infty}^{\infty} f(kx)\,dx = \frac{1}{k}\int_{-\infty}^{\infty} f(x)\,dx$$

4) $$\int_{-\infty}^{\infty} |f(x+h)-f(x)|\,dx \to 0 \text{ as } h\to\infty$$

<u>Pf of 4</u>: Given $\varepsilon>0$, $\exists\, R>0$ such that

$$\int_{|x|>R-1} |f|\,dx < \frac{\varepsilon}{4}.$$ Assume $|h|<1$.

Now $$\int_{-\infty}^{\infty} |f(x+h)-f(x)|\,dx \le$$

$$\int_{|x|>R} |f(x+h)|\,dx + \int_{|x|<R} |f(x)|\,dx$$

$$+\int_{-R}^{R} |f(x+h)-f(x)|\,dx$$

f is uniformly continuous on compact $[-R-1, R+1]$, so $\exists\, \delta>0$ such that $0<\delta<1$ and

$$|f(x+h)-f(x)| < \frac{\varepsilon}{4R} \text{ if } |h|<1 \text{ and } x\in[-R,R].$$

Now $\displaystyle\int_{-\infty}^{\infty} |f(x+h)-f(x)|\,dx <$

$$\underbrace{\int_{|x|>R-1} |f|\,dx}_{<\frac{\varepsilon}{4}} + \underbrace{\int_{|x|>R} |f|\,dx}_{<\frac{\varepsilon}{4}}$$

$$\underbrace{+ \int_{-R}^{R} \underbrace{|f(x+h)-f(x)|}_{<\frac{\varepsilon}{4R}}\,dx}_{<\frac{\varepsilon}{4R}\cdot 2R=\frac{\varepsilon}{2}} \quad \text{if } |h|<1.$$

Note: $\displaystyle\int_{|x|>R} |f(x+h)|\,dx < \int_{|x|>R-1} |f|\,dx < \frac{\varepsilon}{4}$

Schwarz ← German

Schwartz ← French, Laurent Schwartz

$\mathcal{S}(\mathbb{R})$ = fcns that go to zero so fast as $x\to\infty$ or $x\to-\infty$ that $|x^n|\,|f^{(k)}(x)|\to 0$ as $|x|\to\infty$ too for each n,k.

Note: $e^{-x^2}\in\mathcal{S}$

Facts: 1) $f\in\mathcal{S} \Rightarrow f'\in\mathcal{S}$

2) $f\in\mathcal{S} \Rightarrow x f(x)\in\mathcal{S}$

So (polys)$\cdot$(derivatives of f) in $\mathcal{S}$ when $f\in\mathcal{S}$.

3) $f\in\mathcal{S} \Rightarrow \hat{f}\in\mathcal{S}$

Properties of Fourier Transform on $\mathcal{S}$:

$$f(x) \longrightarrow \hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-isx}\,dx$$

1) $f(x+h) \longrightarrow e^{ish}\hat{f}(s)$

2) $f(x)e^{-ixh} \longrightarrow \hat{f}(s+h)$

3) $f(\delta x) \longrightarrow \frac{1}{\delta}\hat{f}\left(\frac{s}{\delta}\right)$

4) $f'(x) \longrightarrow is\hat{f}(s)$

5) $-ixf(x) \longrightarrow \hat{f}'(s)$

Fourier Transform of e^{-x^2}:

$$\widehat{e^{-x^2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2} e^{-isx}\,dx = F(s)$$

$$F'(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}\left[-ix e^{-isx}\right]dx$$

diff under $\int$ sign

$$= \frac{1}{2\pi}\frac{i}{2}\int_{-\infty}^{\infty} \underbrace{\left(-2xe^{-x^2}\right)}_{\frac{d}{dx}\left(e^{-x^2}\right)} e^{-isx}\,dx$$

$$\underset{\text{int by parts}}{\overset{\uparrow}{=}} -\frac{i}{4\pi}\int_{-\infty}^{\infty} e^{-x^2}\underbrace{\frac{d}{dx}\left[e^{-isx}\right]}_{=-is\,e^{-isx}}dx$$

$$= -\frac{1}{2}\cdot s\cdot \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}e^{-isx}\,dx}_{=F(s)\,!}$$

Get first order linear ODE:

$$F'(s)+\tfrac{1}{2}s\,F(s)=0 \leftarrow \text{Int factor } u=e^{\int \frac{1}{2}s\,ds}=e^{s^2/4}$$

$$\underbrace{e^{s^2/4}F'(s)+\tfrac{1}{2}s\,e^{s^2/4}F(s)}=0 \leftarrow \text{mult by } u$$

$$\frac{d}{ds}\Big[\underbrace{e^{s^2/4}F(s)}\Big]\equiv 0$$

must be const $= C$

Get $F(s)=C\,e^{-s^2/4}$

Plug in $s=0$ to get C:

$$F(0)=C\cdot e^0=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}\cdot \underset{e^{-i\cdot 0\cdot s}}{e^0}\,dx=\frac{\sqrt{\pi}}{2\pi}=\frac{1}{2\sqrt{\pi}}$$

So $\boxed{\widehat{e^{-x^2}} = \frac{1}{2\sqrt{\pi}} e^{-s^2/4}}$

Inverse Fourier Transform

$$\check{g}(x) = \int_{-\infty}^{\infty} g(s)\, e^{isx}\, ds$$

Repeat ideas above to show

$$\left(\frac{1}{2\sqrt{\pi}} e^{-s^2/4}\right)^{\vee} = e^{-x^2}$$

Philosophy: Fourier inversion formula true for bump fcn $\begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \\ \varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) \end{cases}$ and translates

$\Longrightarrow$ true for linear combinations.

$$\underbrace{\varphi_\varepsilon * f}_{\downarrow\, f} \approx \sum c_j\, \varphi_\varepsilon(x-t_j)\, f(t_j)$$

$f \longleftarrow$ true for $f \in \mathcal{S}$ by taking limits.

Parseval's Identity (called Plancherel Formula)

$$\|f\|^2 = 2\pi \|\hat{f}\|^2$$

$$\int_{-\infty}^{\infty} |f|^2 \, dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}|^2 \, ds$$

Given $f \in L^2(\mathbb{R})$, take $f_n \in \mathcal{S}$ with $f_n \xrightarrow[L^2]{} f$.

Plancherel $\Rightarrow$ $\hat{f}_n \longrightarrow$ something

call this $\hat{f}$!