Lecture 36 Plancherel identity (~ Parseval's)

Hwk 9 posted later today. Due Mon. (Last one)

Plancherel: $\displaystyle\int_{-\infty}^{\infty} |f|^2\,dx = 2\pi\int_{-\infty}^{\infty} |\hat{f}|^2\,ds$ for $f \in \mathcal{S}$.

$$\|f\|_{L^2} = \sqrt{2\pi}\,\|\hat{f}\|_{L^2}$$

Key: $\displaystyle\int_{-\infty}^{\infty} f\,\hat{g}\,dx = \int_{-\infty}^{\infty} \hat{f}\,g\,dx$

Hmmm. what if $\hat{g} = \bar{f}$?

Then $g = \check{\bar{f}}$

$$= \int_{-\infty}^{\infty} \overline{f(x)}\,\underbrace{e^{isx}}_{\overline{e^{-isx}}}\,dx$$

$$= \overline{2\pi\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-isx}\,dx}$$

$$= 2\pi\,\overline{\hat{f}(s)}$$

$$\int_{-\infty}^{\infty} f\,\underbrace{\hat{g}}_{\bar{f}}\,dx = \int_{-\infty}^{\infty} \hat{f}\,\underbrace{g}_{2\pi\,\overline{\hat{f}}}\,dx$$

$$\int_{-\infty}^{\infty} |f|^2\,dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}|^2\,dx \checkmark$$

Exciting consequence: Fouier transform extends to $L^2(\mathbb{R})$.

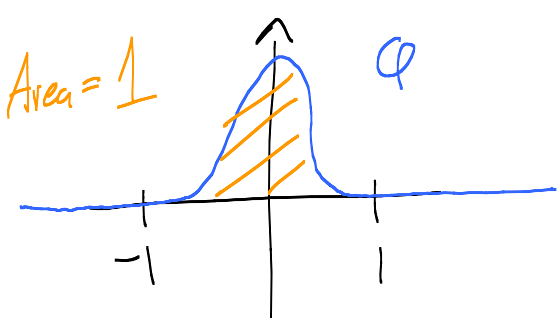
Why: $\underbrace{C_0^\infty(\mathbb{R})}_{\text{dense subspace of } L^2(\mathbb{R})} \subset \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$.

Pf: $f \in L^2(\mathbb{R}) \leftarrow$ Lebesgue measurable fcn such that $\int_{-\infty}^{\infty} |f|^2\,dx < \infty$.

Step 1: $f_N = \chi_{[-N,N]} \cdot f$

Easy fact: $f_N \xrightarrow{L^2} f$

Step 2: Smooth f_N out with a C_0^∞ bump fcn



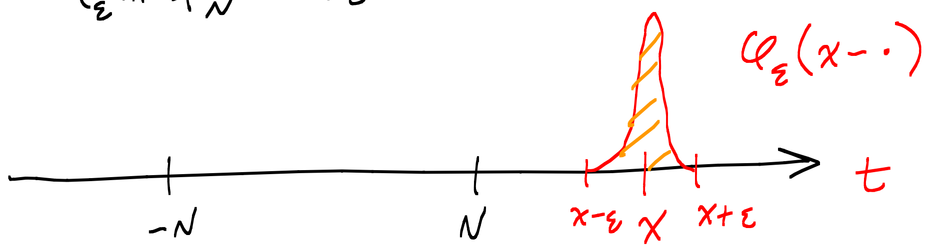
$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$$

Fact: $\underbrace{\varphi_\varepsilon * f_N}\ \longrightarrow f_N$ in L^2.

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(x-t)\, f_N(t)\, dt$$

DQ's under $\int \to \int$(derivative in x)

So $\varphi_\varepsilon * f_N$ is C^∞ smooth.



<u>and</u> it has compact support in $[-N-\varepsilon, N+\varepsilon]$.

<u>Step</u>: Get seq $\psi_n \in C_0^\infty(\mathbb{R})$ with $\psi_n \to f$ in L^2.

<u>Step</u>: Big fact: $L^2(\mathbb{R})$ is a <u>complete</u> metric space.

$\psi_n \to f$ in L^2.

So ψ_n is a Cauchy seq in L^2:

Given $\varepsilon > 0$, $\exists\, N$ such that

$$\underbrace{\|\psi_n - \psi_m\|}_{= \sqrt{2\pi}\,\|\hat{\psi}_n - \hat{\psi}_m\|} < \varepsilon \quad \text{when } n, m > N.$$

Planchere!!

Aha! $\hat{\psi}_n$ are also a Cauchy seq!

So they converge to, say to F.

Aha! F is what $\hat{f}$ <u>must</u> be.

Easy to show $\displaystyle \int \underset{\substack{\uparrow \\ \psi_n}}{f}\ \hat{g}\, dx = \int \underset{\substack{\uparrow \\ \hat{\psi}_n}}{\hat{f}}\ g\, dx$

$$\int |\underset{\substack{\uparrow \\ \psi_n}}{f}|^2\, dx = 2\pi \int |\underset{\substack{\uparrow \\ \hat{\psi}_n}}{\hat{f}}|^2\, dx$$

<u>Ques:</u> $\hat{f}$ is uniquely determined (up to a set of "measure zero"). Fact: $f = g$ in $L^2 \Longrightarrow$

$f(x) = g(x)$ a.e. $\leftarrow$ "almost everywhere"

<u>Heisenberg uncertainty principal</u>: Can't know exactly both the position and momentum of a particle at a specific time.

Equivalent to: f and $\hat{f}$ cannot both have <u>compact</u> support!

Problem: Suppose $f \in C_0^\infty(\mathbb{R})$ supported in $[a,b] \subset (-\pi, \pi)$.

Claim: $\hat{f}$ cannot have compact support.

$$\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

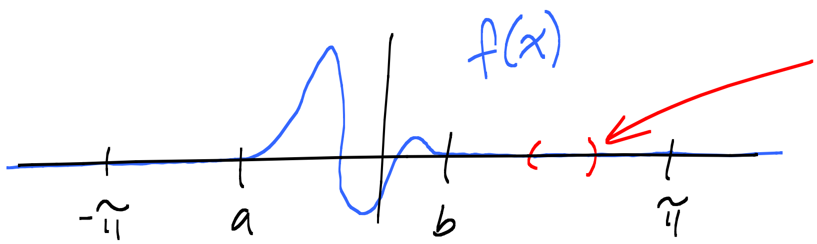
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-isx}\, dx$$

Aha! If $\hat{f}$ is supported in $[-N, N]$, N an integer,

the Fourier coeff c_n for f on $(-\pi, \pi)$

would be zero if $|n| > N$.

What? $$f(x) = \underbrace{\sum_{|n|<N} c_n e^{inx}}_{\text{Trig poly!}} \quad \text{on } (-\pi, \pi)$$



Hmmm. Can a finite sum of sines and cosines vanish on an interval?

Hint: Power series.