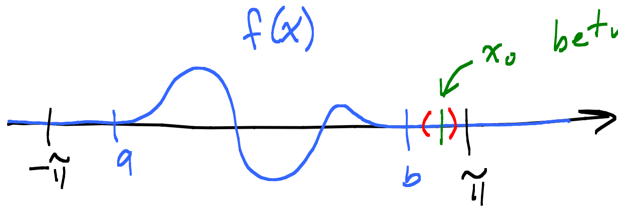
Lecture 37 Heisenberg uncertainty principle

Thm: f and $\hat{f}$ cannot both have compact support.

Why:



between b and π.

What if $\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-isx}\,dx = 0$ for $|s| > N$ ← pos. int.

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-isx}\,dx}_{\text{Fourier series } c_n = 0} = 0 \quad \text{for } s \in \mathbb{Z},\ |s| > N$$

On $[-\pi, \pi]$: $\underbrace{f(x)}_{\text{real}} = \sum_{|n| \le N} c_n e^{inx}$ ← Trig poly.

c_n: complex $\qquad e^{inx}$: $\cos nx + i \sin nx$

$$= \underbrace{\left(\sum_{n=0}^{N} \alpha_n \cos nx + \beta_n \sin nx \right)}_{\text{real part}} + i \underbrace{\left(\text{Imag part} \right)}_{\text{must be zero}}$$

real part ↑ must be $f(x)$ on $(-\pi, \pi)$.

$f(x) = (\text{Real trig poly}) = \text{power series} = \sum_{n=0}^{\infty} A_n (x - x_0)^n$ on $(-\pi, \pi)$

Taylor's: $A_n = \frac{f^{(n)}(x_0)}{n!} = 0$ for all n. Conclude $f \equiv 0$.

What if $[a,b]$ not $\subset (-\pi, \pi)$?

$$\widehat{f(\delta x)} = \frac{1}{\delta}\hat{f}\left(\frac{x}{\delta}\right)$$

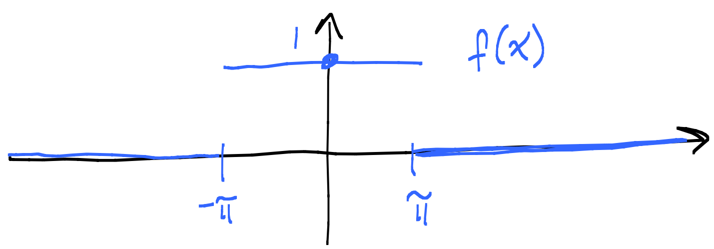
Aha! Take big enough δ that $\frac{1}{\delta}[a,b] \subset [-\pi, \pi]$.

Use previous result.

Or use $[a,b] \subset (-L, L)$ and use Fourier series

$$\sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx.$$

Fourier Transform in MAPLE:



Note: $\hat{f}$ compact supp $\Rightarrow$ f not too.

$f(x) = \check{\hat{f}}$ ← just puts $-s$ where s was in previous arg.

$$\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-isx}\, dx$$

real:

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\cos sx}_{\text{even}} - i\, \underbrace{\sin sx}_{\text{odd}}\, dx$$

$$= \frac{1}{2\pi}\cdot 2\int_{0}^{\pi} \cos sx\, dx$$

$$=$$

complex:

$$= \frac{1}{2\pi}\left[-\frac{1}{is} e^{-isx}\right]_{-\pi}^{\pi}$$

$$= \frac{-1}{2\pi i s}\left[e^{-is\pi} - e^{is\pi}\right]$$

$$= \frac{1}{\pi s}\cdot \frac{e^{is\pi} - e^{-is\pi}}{2i}$$

$$= \frac{\sin \pi s}{\pi s}$$

Fourier Inversion Formula (or Fourier integral):

$$f(x) = \int_{-\infty}^{\infty} \underbrace{\frac{\sin \pi s}{\pi s}}_{\text{even}} \cdot \underbrace{e^{isx}}_{\underbrace{\cos sx}_{\text{even}} + i \underbrace{\sin sx}_{\text{odd}}} \, ds$$

$$= \int_{-\infty}^{\infty} \frac{\sin \pi s}{\pi s} \cos sx \, ds = 2\int_{0}^{\infty} \frac{\sin \pi s \cos sx}{\pi s} \, ds$$

Real Fourier integral:

$$\hat{f}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{real}} \underbrace{e^{-isx}}_{\cos sx - i \sin sx} \, dx$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos sx \, dx}_{\substack{A(s) \\ \text{even!}}} - i \underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin sx \, dx}_{\substack{B(s) \\ \text{odd}}} \right]$$

$$\underbrace{f(x)}_{\text{real}} = \int_{-\infty}^{\infty} \underbrace{\hat{f}(s)}_{\frac{1}{2}(A - iB)} \, \underbrace{e^{isx}}_{\cos sx + i \sin sx} \, ds$$

even, even, odd, odd

$$= \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} \underbrace{A(s) \cos sx}_{\text{even}} + \underbrace{B(s) \sin sx}_{\text{even}} \, ds}_{\text{real part}}$$

$$+ \, i \, \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{\underbrace{\overset{\text{even}}{A(s)} \overset{\text{odd}}{\sin sx}}_{\text{odd}} - \underbrace{\overset{\text{odd}}{B(s)} \overset{\text{even}}{\cos sx}}_{\text{odd}} \, ds}_{=0}$$

$$f(x) = \int_{0}^{\infty} A(s) \cos sx + B(s) \sin sx \, ds$$

where

$$A(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos sx \, dx$$

$$B(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin sx \, dx$$