Lecture 38 Dirichlet problem on UHP

(HWK 9 due Mon.)

What is a set of "measure zero" $S' \subset \mathbb{R}$?

[Given $\varepsilon > 0$, can cover S' by open intervals (a_n, b_n) such that $\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon$.

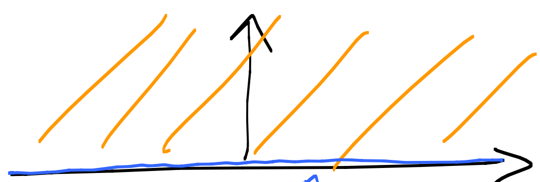
$\mathbb{Q} \subset \mathbb{R}$ is a set of measure zero.

Why: enumerate $\mathbb{Q}$: $\{q_n\}_{n=1}^{\infty}$. Given $\varepsilon > 0$,

cover $\mathbb{Q}$ by $\left(q_n - \frac{\varepsilon}{2^n}, q_n + \frac{\varepsilon}{2^n} \right)$.

$$\left(\sum_{0}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \right)$$

D. Prob on UHP:



Given $f(x)$ on $\mathbb{R}$. Assume $f \in \mathcal{S}$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

steady state heat prob

$$u(x,0) = f(x)$$

Try $u(x,y) = X(x)Y(y)$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$$

Hmmm. Write $\lambda = \pm k^2$

$Z'' \pm k^2 Z = 0$ in t

$-k^2$: Z: e^{-kt}, e^{kt}

$+k^2$: Z: $\cos kx, \sin kx$

or e^{ikx}, e^{-ikx}

Physical consideration: Can't have either e^{-kx} or e^{kx} in $X(x)$ because sol^{n}s can't blow up.

Get $X(x) = e^{isx}$, $Y(y) = e^{-|s|y}$

Get $u_s(x,y) = e^{isx} e^{-|s|y}$ $\quad s \in \mathbb{R}$

Note: $s=0$ is constant sol^n. $\lambda = 0$. ← $A t + B$

need $A = 0$.

Big idea: "Add up" these basic sol^{n}s:

$$\sum_{s \in \text{grid}} c_s e^{isx} e^{-|s|y}$$ ← Aha! Looks like a Riemann sum

$$u(x,y) = \int_{-\infty}^{\infty} g(s) e^{isx} e^{-|s|y} \, ds$$

Hope: $\Delta u = \Delta \int = \int \Delta = 0$

$$u(x,0) = \int_{-\infty}^{\infty} g(s) e^{isx} \, ds \overset{\text{want}}{=} f(x)$$

Aha! Need $g(s) = \hat{f}(s)$.

Get $$\boxed{u(x,y) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx} e^{-|s|y}\, ds}$$

This works!

$$u(x,y) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt \right) e^{isx} e^{-|s|y}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} e^{-ist} e^{isx} \underbrace{e^{-|s|y}}\, ds \quad dt$$

$y>0$ makes this ok.

Poisson kernel for UHP is

$$\int_{-\infty}^{\infty} e^{is(x-t)-|s|y}\, ds$$

$$= \underbrace{\int_{-\infty}^{0}} + \underbrace{\int_{0}^{\infty}}$$

$$= \vdots \qquad\qquad = \frac{1}{i(x-t)-y} e^{(i(x-t)-y)s} \Big|_{s=0}^{\infty}$$

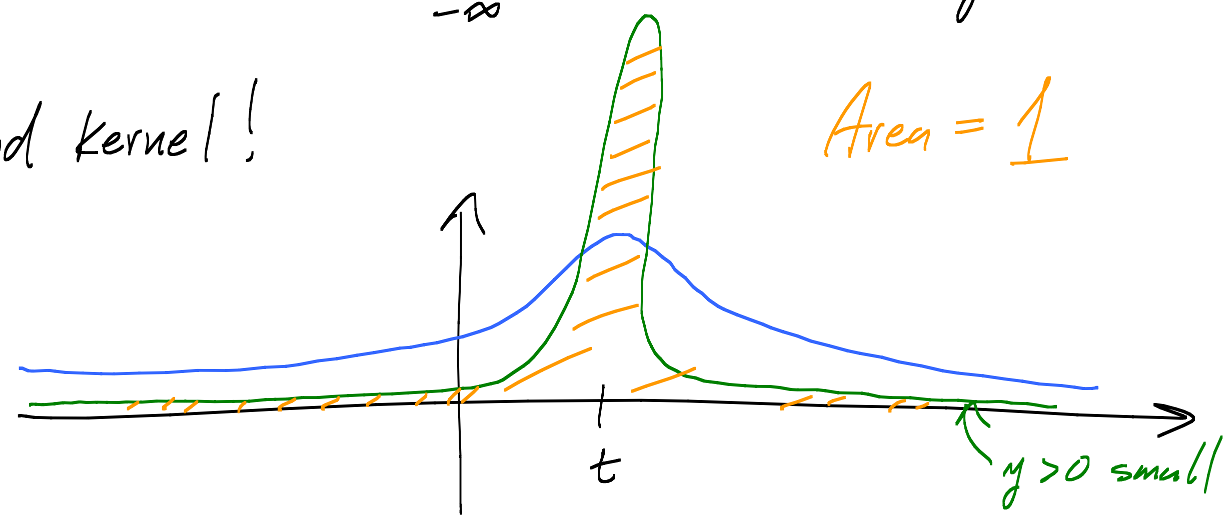
$$= \frac{1}{i(x-t)+y} \qquad\qquad = \frac{-1}{i(x-t)-y}$$

$$= \frac{2y}{(x-t)^2 + y^2}$$

Poisson formula:

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2+y^2}\, dt$$

Good kernel!



$$P(x,y,t) = \frac{1}{\pi} \frac{y}{(x-t)^2+y^2}$$

1) $P > 0 \quad (y>0)$,

2) $\int_{-\infty}^{\infty} P(x,y,t)\, dt = 1$ for all $y>0$,

3) $P(x,y,t) \to 0$ unif on $\mathbb{R} - [t-\delta, t+\delta]$

as $y \searrow 0$ for any fixed δ.

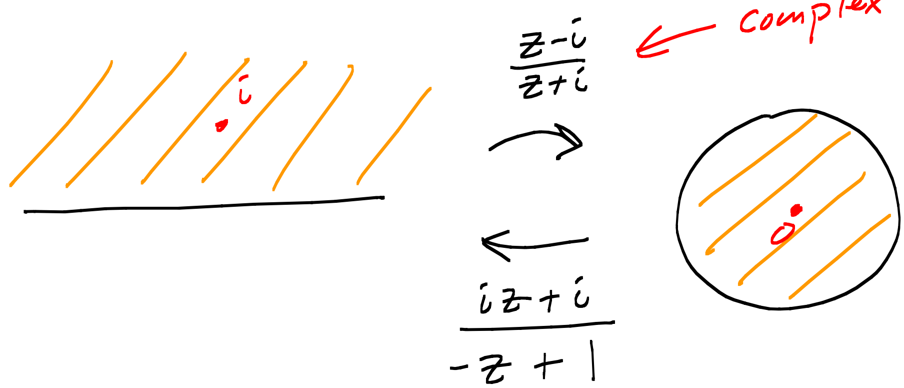
4) $\Delta_{x,y} P \equiv 0.$

$1-4 \Rightarrow$ Formula solves D. Prob. ✓

Forget how you got it.

Take limits of fcns in $\mathcal{A}$ to get more gen'l result.

<u>Facts:</u>



← complex diff'ble "analytic"

Analytic fcns preserve harmonic fcns.

<u>Another approach to UHP prob:</u>

Take F. T. in x-variable:

$$\hat{u}(s,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,y)\, e^{-isx}\, dx$$