Lecture 40 Convolution, Legendre polynomials, review

Final Exam Tuesday eve 7:00 – 9:00 pm in UNIV 003 (here)

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt$$

Facts: $f, g \in \mathcal{S}(\mathbb{R})$. Then

1) $\widehat{(fg)} = 2\pi\, \hat{f} * \hat{g}$

2) $\widehat{f*g} = \hat{f} \cdot \hat{g}$

true more generally by taking limits

Pf: Simple change of vars plus Fubinni.

Note: $*$ for Fourier transform is different than $*$ for Laplace trasform.

For $\mathcal{L}$, the Laplace transf: $(f*g)(x) = \int_0^x f(x-t)\,g(t)\,dt$

$$\mathcal{L}[f*g] = \mathcal{L}[f]\cdot\mathcal{L}[g]$$

Legendre polynomials $\quad p_n(x) = \dfrac{d^n}{dx^n}\underbrace{(x^2-1)^n}_{\text{poly deg } 2n} \leftarrow$ poly deg n

Easy fact: Linear span $\{p_0, p_1, p_2, \ldots, p_N\} = \left(\text{polys of deg } N\right)$

Legendre Eqn:

$$\underbrace{(1-x^2)}\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0$$

bad things happen sol^{n}s when $x = \pm 1$

Origin: Vibrating bowling ball or hot bowling ball, steady state or more generally:

Separation of vars: $u(\rho,\theta,\psi) = P(\rho)\,\Theta(\theta)\,\Psi(\psi)$

Get λ.

ODE in P, ρ is the Legendre eqn!

Singular Sturm-Liouville problem.

Only for special λ do you get one sol^n that is nice at $\rho = 1$. The Legendre polys are the nice one.

n-th one: $$p_n(x) = \frac{d^n}{dx^n}(x^2-1)^n$$ alt being even and odd!

Facts: $$\int_{-1}^{1} p_n(x)\, f(x)\, dx = (-1)^n \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\, dx$$

(Int by parts n-times)

See $p_n \perp x^m$ when $m < n$.

So $p_n \perp p_m$ when $m < n$.

$$\|p_n\|^2 = \int_{-1}^{1} p_n(x)^2\,dx = (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n\,dx$$

$$= \frac{(n!)^2\, 2^{n+1}}{2n+1}$$

Legendre expansion:

$$f \text{ "=" } \sum_{n=0}^{\infty} c_n p_n$$

What is c_n?

$$\int_{-1}^{1} f \cdot p_m\,dx = \sum_{n=0}^{\infty} c_n \underbrace{\int_{-1}^{1} p_n p_m\,dx}_{\substack{=0 \\ n \neq m}}$$

$$= c_m \int_{-1}^{1} p_m^2\,dx$$

So

$$c_m = \frac{\int_{-1}^{1} f p_m\,dx}{\int_{-1}^{1} p_m^2\,dx \leftarrow A_m}$$

<u>Bessel's ineq:</u>

$$f = \sum b_n \varphi_n \leftarrow \varphi_n \text{ orthonormal}$$

$$\int_{-1}^{1} \varphi_n^2\,dx = 1$$

$$\sum |b_n|^2 \leq \int_{-1}^{1} |f|^2\,dx$$

Make p_n's orthonormal via: $\varphi_n = \dfrac{p_n}{\sqrt{\int_{-1}^{1} p_n^2\,dx}} = \dfrac{1}{\sqrt{A_n}}\, p_n$

Now $f = \sum c_n P_n = \sum \underbrace{c_n \sqrt{A_n}}_{b_n} \underbrace{\frac{P_n}{\sqrt{A_n}}}_{\varphi_n}$

Bessel's: $\displaystyle\sum_{n=0}^{\infty} A_n c_n^2 \le \int_{-1}^{1} f^2\,dx$

Why is Bessel's = Parseval's identity for Legendre polys?

Key: Weierstraß says polys "dense" among continuous fcns on $[-1,1]$, so given continuous f on $[-1,1]$ and $\varepsilon > 0$, there exists a linear combo of Legendre polys $p = \sum_{n=0}^{N} B_n P_n$ with $|p - f| < \varepsilon$ on $[-1,1]$. Note: $\displaystyle\int_{-1}^{1} |p-f|^2\,dx \le \underbrace{\varepsilon^2\big(1-(-1)\big)}_{2\varepsilon^2}$

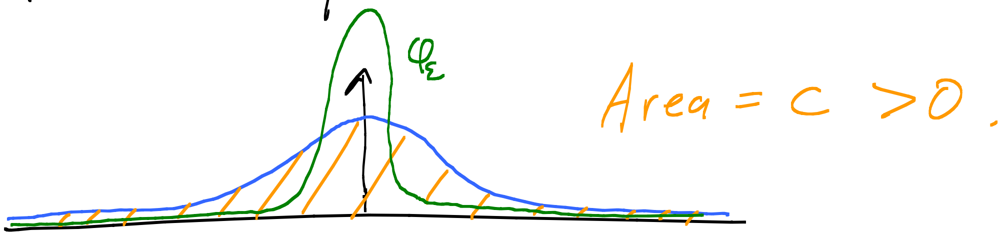
Get Parseval's like this

$$\underbrace{0 \le \int_{-1}^{1} f^2\,dx - \sum_{0}^{N} b_n^2}_{} = \int_{-1}^{1} \Big| f - \sum_{0}^{N} b_n \varphi_n \Big|^2 dx \le \underbrace{\int_{-1}^{1} \Big| f - \sum_{0}^{N} B_n P_n \Big|^2 dx}_{\le 2\varepsilon^2}$$

the "Fourier" coeff

Aha! this $\to 0$ as $N \to \infty$.

Bump fcn review prob. e^{-x^4}



Let $\varphi(x) = \frac{1}{c} e^{-x^4}$ to get unit area

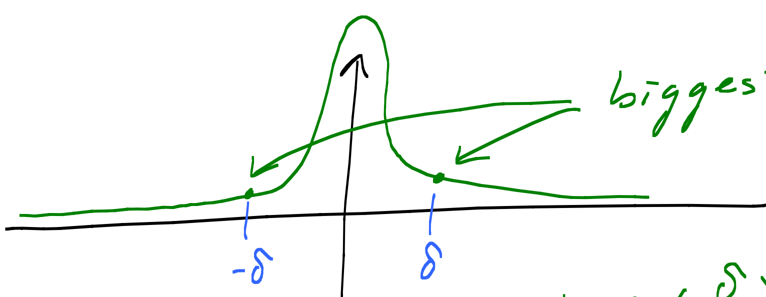
$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ is a bump fcn!

1) $\varphi_\varepsilon > 0$

2) $\int_{-\infty}^{\infty} \varphi_\varepsilon \, dx = 1$

3) $\varphi_\varepsilon(x) \to 0$ uniformly on $(-\infty, -\delta] \cup [\delta, \infty)$

as $\varepsilon \to 0$



biggest φ_ε gets on

$\frac{1}{\varepsilon}\varphi\left(\frac{\delta}{\varepsilon}\right) =$

$\frac{1}{\varepsilon}\frac{1}{c} e^{-(\delta/\varepsilon)^4}$

$\longrightarrow 0$ as $\varepsilon \to 0$.

Not as good as e^{-x^2} because $\widehat{e^{-x^2}} = \frac{1}{2\sqrt{\pi}} e^{-s^2/4}$ ← bump fcn!