Review for the midterm exam (on Mon., March 4 in class)

1. Show that the three functions x^7, $\cos 3x$, and 1 are pairwise orthogonal on $[-\pi, \pi]$. If

$$f(x) = c_0 + c_1 x^7 + c_2 \cos 3x,$$

write an integral formula for c_1 involving $f(x)$ and the given functions.

$$\langle x^7, \cos 3x \rangle = \int_{-\pi}^{\pi} \underbrace{\underbrace{x^7}_{\text{odd}} \; \underbrace{\cos 3x}_{\text{even}}}_{\text{odd}} \, dx = 0 \checkmark$$

want

symmetric interval

fact: f odd all cosine terms in F.S. = 0

$$\langle 1, x^7 \rangle = 0 \checkmark \text{ (same)}$$

$$\langle 1, \cos 3x \rangle = \int_{-\pi}^{\pi} \cos 3x \, dx = \frac{1}{3} \sin 3x \Big|_{-\pi}^{\pi} = 0 - 0 \checkmark$$

$$\int_{-\pi}^{\pi} : \quad f(x) \cdot x^7 = c_0 \cdot 1 \cdot x^7 + c_1 x^7 \cdot x^7 + c_2 \cos 3x \cdot x^7$$

$$\int_{-\pi}^{\pi} f(x) x^7 \, dx = 0 + c_1 \int_{-\pi}^{\pi} x^{14} \, dx + 0$$

$$= 2c_1 \int_0^{\pi} x^{14} \, dx = 2c_1 \frac{\pi^{15}}{15}$$

Solve for c_1.

Normalize: $\|x^7\| = \left(\int_{-\pi}^{\pi} (x^7)^2 \, dx \right)^{1/2} = \left(\frac{2\pi^{15}}{15} \right)^{1/2} = A$

$\varphi_2 = \frac{x^7}{A}$ has L^2 norm <u>one</u>.

$$S_N = 1 + \cos\theta + \cos 2\theta + \cdots + \cos N\theta.$$

$$\cos n\theta = \mathrm{Re}\left[e^{in\theta}\right] = \mathrm{Re}\left[\left(e^{i\theta}\right)^n\right]$$

$$1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}$$

$$S_N = \mathrm{Re}\left[1 + \left(e^{i\theta}\right) + \left(e^{i\theta}\right)^2 + \cdots + \left(e^{i\theta}\right)^N\right] \leftarrow z = e^{i\theta}$$

$$= \mathrm{Re}\left[\frac{1 - \left(e^{i\theta}\right)^{N+1}}{1 - e^{i\theta}}\right]$$

$$= \mathrm{Re}\left[\frac{1 - e^{i(N+1)\theta}}{1 - e^{i\theta}} \cdot \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}}\right]$$

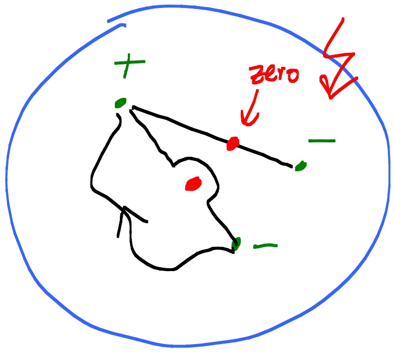
makes denom real

Or:

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$$

$$S_N = \left[\frac{1}{2}\sum_{n=0}^{N}\left(e^{i\theta}\right)^n + \frac{1}{2}\sum_{n=0}^{N}\left(e^{-i\theta}\right)^n\right]$$

4. Prove that a continous real valued function on an open disc that has a single zero at the center of the disc must be either always positive away from the zero, or always negative. Use this fact to show that a continuous function that satisfies the averaging property cannot have an isolated zero. Why does this imply that harmonic functions cannot have isolated zeroes?



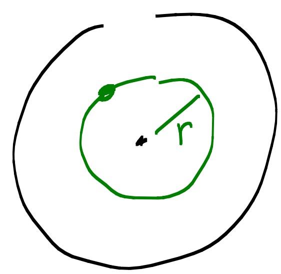
Intermediate Value Thm!

$f(z(t))$

If u is harmonic, isolated zero at 0, then

ave property yields:

$$0 = u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{u(re^{i\theta})}_{\text{always + or always −}}\, d\theta$$



can't be zero!

1. Suppose that $f(x)$ is a C^1-smooth function on $[a,b]$. Use integration by parts to show that

$$I = \int_a^b \underbrace{f(x)}_{u} \underbrace{\cos(Nx)\, dx}_{dv}$$

$du = f'(x)\,dx$

$v = \frac{1}{N} \sin Nx$

tends to zero as the positive integer N tends to infinity.

$$I = uv\Big|_a^b - \int_a^b v\, du = \frac{1}{N} f(x) \sin Nx - \int_a^b \frac{1}{N} \sin Nx\, f'(x)\,dx$$

$$= \frac{1}{N} \left[\underbrace{f(b)\sin Nb - f(a)\sin Na}_{A} - \int_a^b f'(x) \sin Nx\, dx \right]$$

$$\left| \int_a^b \right| \le \underbrace{\left(\max_{[a,b]} |f'| \cdot 1 \right) \cdot (b-a)}_{M}$$

So $\quad |I| \le \frac{1}{N} \left(|A| + M \right) \to 0 \quad$ as $\quad N \to \infty.$

2. Estimate the integral in problem one in terms of the maximum value M of $|f(x)|$ on $[a, b]$.

$$\left| \int_a^b f(x) \cos Nx \, dx \right| \leq \int_a^b \underbrace{|f(x)|}_{\leq M} \cdot \underbrace{|\cos Nx|}_{\leq 1} \, dx$$

$$\leq \int_a^b M \, dx = M(b-a)$$

3. Show that the integral in problem one goes to zero as $N \to \infty$ when f is merely assumed to be continuous by approximating f by a polynomial and then using your results from problem one and two.

Let $\varepsilon > 0$. Weierstraß theorem says there is a poly P with $|f(x) - P(x)| < \frac{\varepsilon}{2} \cdot \frac{1}{b-a}$.

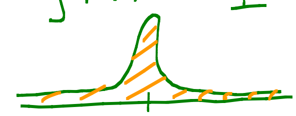
$$\int_a^b f(x) \cos Nx \, dx = \underbrace{\int_a^b \big(f(x) - P(x)\big) \cos Nx \, dx}_{\left|\int_a^b\right| \leq \left(\frac{\varepsilon}{2} \cdot \frac{1}{b-a}\right) \cdot (b-a)} + \underbrace{\int_a^b P(x) \cos Nx \, dx}_{\text{Can make } \left|\int_a^b\right| < \frac{\varepsilon}{2}}$$

by choosing $N >$ some N_0 by prob. 1.

4. Suppose $P(x)$ is a positive continuous function on $[0,1]$ such that $P(1/2) = 10^4$, and $P(x) < {}^{1}\!/100$ when x is in $[0,1/4] \cup [3/4,1]$, and

$P(x) > 0$

$\int P\,dx = 1$



$$\int_0^1 P(x)\,dx = 1.$$

If f is a continuous function such that $|f(x)| < 17$ on $[0,1]$, but $|f(x)| < 1/3$ on $[1/4,3/4]$, estimate how large

$$\left|\int_0^1 P(x)f(x)\,dx\right| = \underbrace{\int_{[0,\frac14]\cup[\frac34,1]}}_{I_1} + \underbrace{\int_{1/4}^{3/4}}_{I_2}$$

could be. What is your best estimate for the bound if the assumption that P is positive is dropped?

$$|I_1| \le \int_0^{1/4} \underbrace{|P(x)|}_{\substack{P(x)\\ < \frac{1}{100}}}\ \underbrace{|f(x)|}_{<17}\,dx + \int_{3/4}^{1} \sim\ < \left(\frac{17}{100}\cdot\frac14\right)\cdot 2$$

$$|I_2| \le \int_{1/4}^{3/4} \underbrace{|P(x)|}_{P(x)}\ \underbrace{|f(x)|}_{<\frac13}\,dx \le \frac13 \int_{1/4}^{3/4} P(x)\,dx$$

$$\le \frac13 \underbrace{\int_0^1 P(x)\,dx}_{=1}$$

Get <u>no</u> estimate if drop $P(x) > 0$ assumption.