Lesson 1 MA 428 Fourier analysis

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HWK	100 points
2 Midterms	200
Final Exm	200
	500 pts

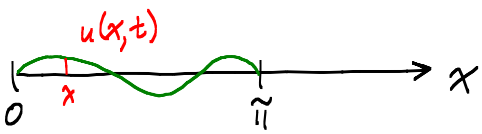
Power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Fourier series: $$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos nx + B_n \sin nx \right)$$

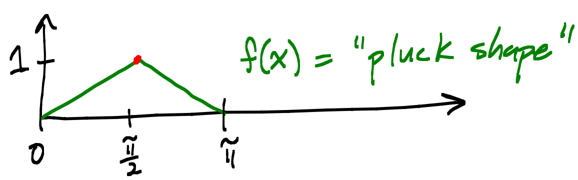
Bernoulli v.s. Bernoulli : Vibrating string problem : Harpsichord



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Time $t=0$



Initial conditions:

$$u(x,0) = f(x)$$

pluck from rest $\longrightarrow$ $$\frac{\partial u}{\partial t}(x,0) = 0$$

Jacob's solution: $$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right]$$ Winner!

Johann's sol^n: Try $u(x,t) = X(x)T(t)$

Assume $c=1$. Plug into PDE:

$$\underbrace{\left[X(x)T''(t) \right]}_{\frac{\partial^2 u}{\partial t^2}} \overset{\text{want}}{=} \underbrace{\left[X''(x)T(t) \right]}_{\frac{\partial^2 u}{\partial x^2}}$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

First let $x=0$.
See right side is const.
Now let x vary.
See left is <u>same</u> const

Aha! Get ODE's: $X'' - \lambda X = 0$, $T'' - \lambda T = 0$.

Boundary conditions: $\begin{cases} u(0,t) = 0 \\ u(\pi, t) = 0 \end{cases}$ for all t.

Hmmm. Need $u(0,t) = X(0)\underbrace{T(t)} = 0$

don't want $T(t) \equiv 0$. Then $u \equiv 0$.

Must have $X(0) = 0$

$u(\pi,t) = \underbrace{X(\pi)}T(t) = 0$.

Need $X(\pi) = 0$.

<u>New problem:</u> $X'' - \lambda X = 0$ (Sturm-Liouville prob.)

Boundary conditions $\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$

ODE: $X'' - \lambda X = 0$ MA 262: Try $X(x) = e^{rx}$

$X'(x) = re^{rx}$

$X''(x) = r^2e^{rx}$

Need: $r^2 e^{rx} - \lambda e^{rx} = 0$ ← want

$(r^2 - \lambda) e^{rx} = 0$ e^{rx} is <u>never</u> zero

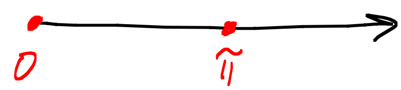
Need $r^2 - \lambda = 0$.

<u>Case $\lambda = 0$:</u> $X'' \equiv 0$

So $X' = c_1$

So $X(x) = c_1 x + c_2$ ← line

$X(x) \equiv 0$ only sol^n with B.C.



<u>Case $\lambda > 0$:</u> $r^2 - \lambda = 0$

$r = \pm\sqrt{\lambda}$ ← let $\lambda = k^2$. $\lambda = \pm k$

Get two sol^{n}s: $X_1(x) = e^{kx}$, $X_2(x) = e^{-kx}$

ODE is <u>linear</u>. So $X(x) = c_1 e^{kx} + c_2 e^{-kx}$ are sol^{n}s too.

Q: Are these <u>all</u> sol^{n}s? E & U Thm says: Given intial conditions $\begin{cases} X(0) = A \\ X'(0) = B \end{cases}$, there <u>exists</u> a sol^n with these I.C.

and it is <u>unique</u>.

Q: Given A, B, can I find $X = c_1 e^{kx} + c_2 e^{-kx}$ with

$\left(X' = c_1 k e^{x} - c_2 k e^{-kx} \right)$

$X(0) = c_1 + c_2 \overset{\text{want}}{=} A$

$X'(0) = kc_1 - kc_2 = B$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ k & -k \end{bmatrix}}_{\text{Need } \det \neq 0} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$\text{Det} = -k - k = -2k \neq 0$ ✓

Can solve ODE with any and all I.C.'s, So we do have the gen^{l} sol^{n}.

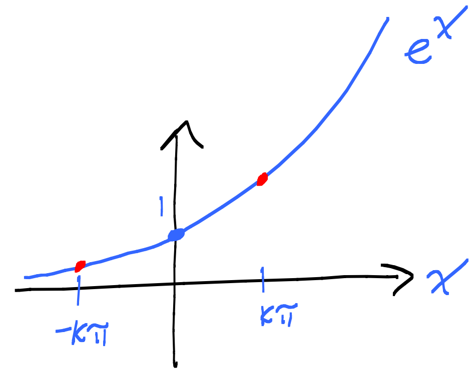
Back to B.C.'s : $\begin{cases} X(0) = c_1 e^{k\cdot 0} + c_2 e^{-k\cdot 0} = 0 & \text{(A)} \\ X(\pi) = c_1 e^{k\pi} + c_2 e^{-k\pi} = 0 & \text{(B)} \end{cases}$

want

(A) $c_1 + c_2 = 0$ so $c_2 = -c_1$.

(B) $c_1 e^{k\pi} + (-c_1) e^{-k\pi} = 0$

$$c_1 \underbrace{(\underset{>1}{e^{k\pi}} - \underset{<1}{e^{-k\pi}})}_{>0} = 0$$



Ouch! Must have $c_1 = 0$. $c_2 = -c_1 = 0$. Get $X(x) \equiv 0$.

<u>Case $\lambda < 0$</u> : $\lambda = -k^2$

$$r^2 - \lambda = 0$$
$$r^2 + k^2 = 0$$
$$r = \pm k i$$

Euler: $e^{ikx} = \cos kx + i \sin kx$ is a <u>complex sol^{n}</u>.

Check it. ✓ $\quad L(\underset{u+iv}{X}) = X'' + k^2 X = 0$

$$L[u] + i L[v] = 0$$

Aha! Real & Imag parts are <u>real</u> sol^{n}s.

Get $\quad X(x) = c_1 \cos kx + c_2 \sin kx$

Check : Wronskian determinant $\neq 0$ ✓

Get gen^l sol^n.

Big question: Can I get non $\equiv 0$ sol^n's of this form satisfying B.C.'s ? Yes!