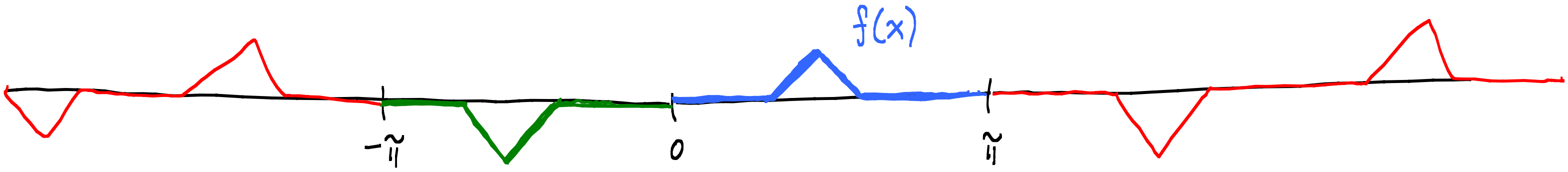
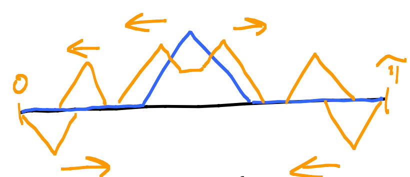
Lesson 4 Fourier Sine series

Hwk 1 due Fri

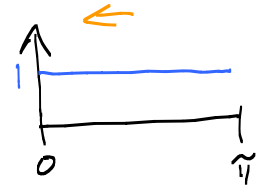
Jacob's harpsichord solⁿ $u(x,t) = \frac{1}{2}\left[\underset{\text{moves left}}{f^*(x+ct)} + \underset{\text{moves right}}{f^*(x-ct)} \right]$



f^* 2π-periodic odd extension of $f(x)$

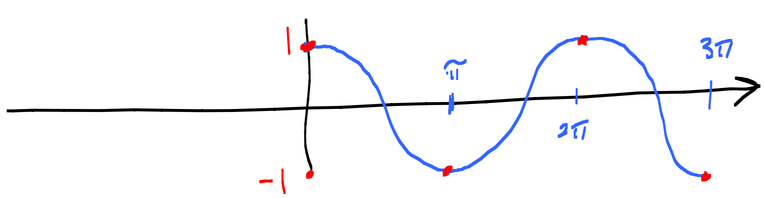


Prob: Find Fourier sine series for



$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{n\pi}\Big[-\underbrace{\cos n\pi}_{(-1)^n} + \underbrace{\cos 0}_{1}\Big]$$



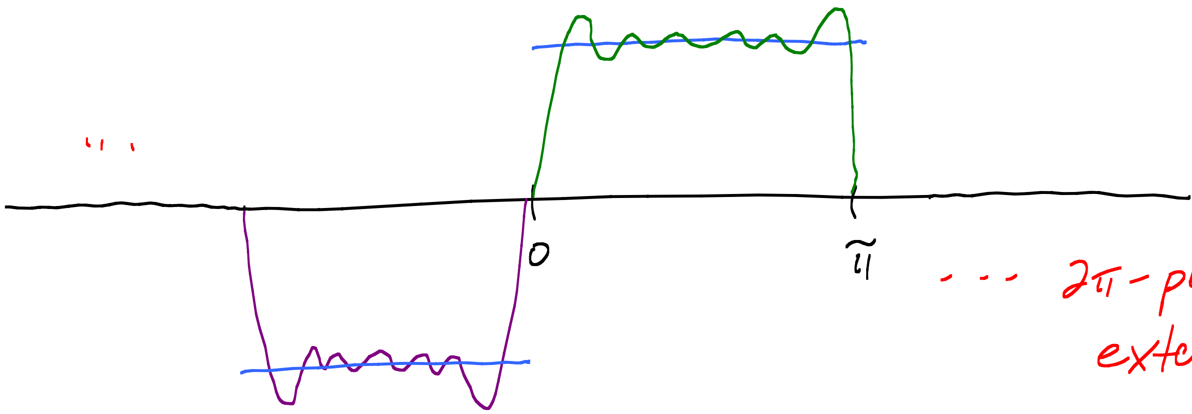
$$A_n = \frac{4}{n\pi} \quad n \text{ odd}, \qquad 0 \quad n \text{ even}$$

$$\sum_1^{\infty} A_n \sin x = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

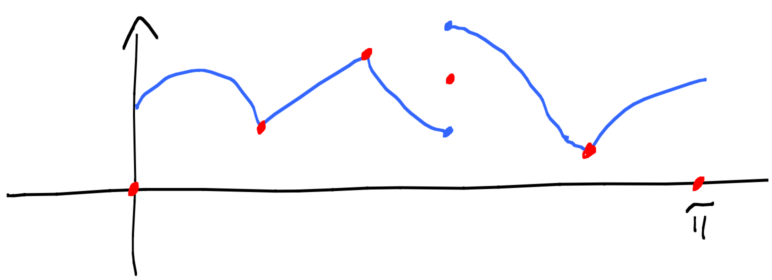
Ouch! $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots = \infty$

Can't use Weierstraß M-test to conclude that the series converges.

<u>Fact</u>: It does converge. (slowly)



<u>Big fact</u>: Fourier sine series converges to $f(x)$ where f is C^1-smooth in $(0,\pi)$, and converges to $f(x)$ more slowly where f has a "corner", and converges to the <u>midpoint</u> of jumps at jumps. (It converges to 0 at 0 and π.)



<u>Full Fourier series</u>:
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

on $[-\pi,\pi]$.

<u>Key</u>: $\{1, \cos nx, \sin nx\}$ $n = 1, 2, 3, \ldots$

are <u>orthogonal</u>!

$$\int_{-\pi}^{\pi} 1 \cdot \underbrace{\sin nx}_{\text{or } \cos nx} \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos nx \, \sin mx \, dx = 0 \qquad \forall\, n, m$$

$$\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx = 0 \qquad n \neq m$$

$n = m$: $$\int_{-\pi}^{\pi} \underbrace{\sin^2 nx}_{\text{or } \cos^2 nx} \, dx = \pi$$

$n = 0$: $$\int_{-\pi}^{\pi} 1^2 \, dx = 2\pi$$

What do a_n's, b_n's <u>have to</u> be?

Hmmm.
$$\int_{-\pi}^{\pi} f(x) \cos 3x \, dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) \cos 3x \, dx$$

$$= 0 + a_3 \underbrace{\int_{-\pi}^{\pi} \cos^2 3x \, dx}_{=\pi} + 0 + \cdots + 0$$

Get $a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 3x\,dx$.

Famous formulas: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ ← Average of f on $[-\pi,\pi]$!

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

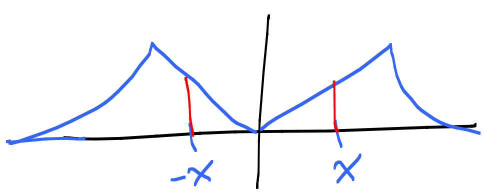
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Fact: If f is odd, all a_n terms vanish. F. Sine series

If f is even, all b_n terms vanish. F. Cosine series

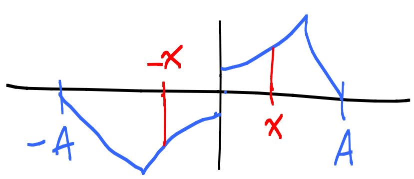
Why: $\displaystyle\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}}\ \underbrace{\cos nx}_{\text{even}}}_{\text{odd}}\,dx = 0$ on $[-A, A]$

Even:



$f(-x) = f(x)$

Odd:

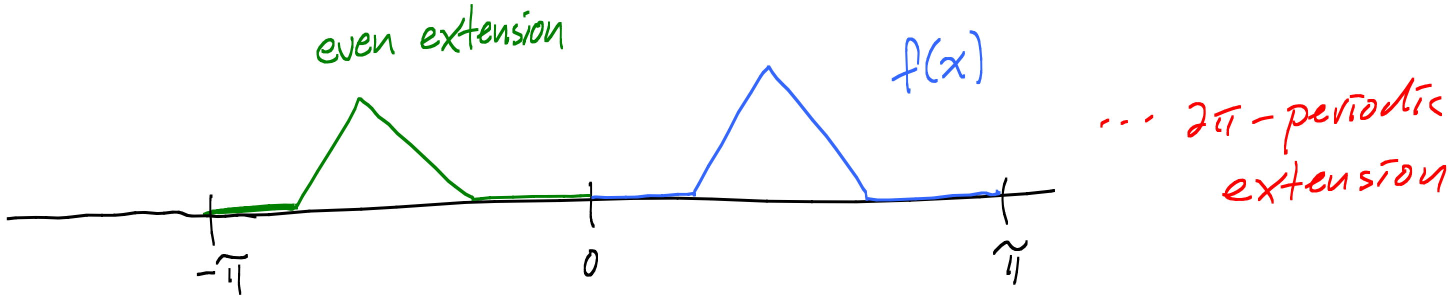


$f(-x) = -f(x)$ areas cancel when interval is symmetric about origin.

Facts: odd · even = odd
odd · odd = even
even · even = even

Fact: $\displaystyle\int_{-A}^{A} \text{even}\, dx = 2\int_{0}^{A} \text{even}\, dx$

Fourier cosine series on $[0,\pi]$.



$$\text{Full Fourier series} = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos nx}_{\text{Fourier cosine series}} + \begin{pmatrix}\text{Sine terms all}\\ \text{zero}\end{pmatrix}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\cdot 2\cdot\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \leftarrow \text{Ave}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}}\underbrace{\cos nx}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Ugh! Too many formulas!

Complex notation:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\Big(a_n \underbrace{\cos nx}_{\frac{e^{inx}+e^{-inx}}{2}} + b_n \underbrace{\sin nx}_{\frac{e^{inx}-e^{-inx}}{2i}}\Big) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \qquad \text{Wow!}$$

Convergence of F. Cosine series : No jumps at $0, \pi$