Lesson 6 Bessel's inequality

$$a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=-N}^{N} c_n e^{inx}$$

(Trigonometric polynomial of degree N)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix} dx$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$e^{-inx} = \overline{e^{inx}}$$

Real inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x) g(x)\,dx$

Complex $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$

Why call it a trig poly? $e^{inx} = (e^{ix})^n$ $\quad e^{-inx} = (e^{-ix})^n$

Bessel's inequality: Assume φ_n are cont. orthonormal real valued fcns on $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \varphi_n(x)\varphi_m(x)\,dx = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \varphi_n(x)^2\,dx = 1 \qquad n = m$$

Write $\langle \varphi_n, \varphi_m\rangle = \int_{-\pi}^{\pi} \varphi_n(x)\varphi_m(x)\,dx$

$$\|f\|^2 = \int_{-\pi}^{\pi} f(x)^2\,dx \qquad \left(\|\varphi_n\| = 1, \ \forall n\right)$$

Hope: $f = \sum_{n=1}^{\infty} c_n \varphi_n$ where $c_n = \langle f, \varphi_n\rangle$.

$$\int_{-\pi}^{\pi} \varphi_m f\,dx = \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} c_n \varphi_n\right) \cdot \varphi_m\,dx = \sum_{1}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} \varphi_n \varphi_m\,dx}_{\text{all but } n=m \ =0}$$

$$= c_m \cdot 1$$

Fact: Let $c_n = \langle f, \varphi_n \rangle$. Then $\sum_{n=1}^{N} c_n \varphi_n$ is the best L^2 approximation to f among fcns of form $\sum_{n=1}^{N} A_n \varphi_n$, meaning

$$\left\| f - \sum_{n=1}^{N} c_n \varphi_n \right\| \le \left\| f - \sum_{n=1}^{N} A_n \varphi_n \right\|$$

with equality only when $c_n = A_n$ for $\forall n$.

Furthermore, $\displaystyle\sum_{n=1}^{N} c_n^2 \le \|f\|^2$.

Aha! $\displaystyle\sum_{n=1}^{\infty} c_n^2 \le \|f\|^2$

Thm: $c_n \to 0$ as $n \to \infty$.

Pf: $\overset{E}{E^*} = \left\| f - \sum_{1}^{N} \overset{A_n}{c_n} \varphi_n \right\|^2 =$ Note: ≥ 0

$$= \int_{-\pi}^{\pi} \left(f - \sum_{n=1}^{N} \overset{A_n}{c_n} \varphi_n \right)\left(f - \sum_{m=1}^{N} \overset{A_m}{c_m} \varphi_m \right) dx$$

$$= \int_{-\pi}^{\pi} f(x)^2\, dx - \sum_{n=1}^{N} \overset{A_n}{c_n} \underbrace{\int_{-\pi}^{\pi} f \varphi_n \, dx}_{c_n} - \sum_{m=1}^{N} \overset{A_m}{c_m} \underbrace{\int f \varphi_m \, dx}_{c_m}$$

$$\sum_{n=1}^{N}\sum_{m=1}^{N} \overset{A_n}{c_n}\,\overset{A_m}{c_m} \underbrace{\int_{-\pi}^{\pi} \varphi_n \varphi_m \, dx}_{\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}}$$

$$= \|f\|^2 - \underbrace{\left(\sum_{n=1}^{N} c_n^2\right)}_{S'} - S' + S'$$

Get $\quad \|f\|^2 - S' \geq 0!$

So $\quad S' \leq \|f\|^2$

$$\sum_{n=1}^{N} c_n^2 \leq \int_{-\pi}^{\pi} f(x)^2 \, dx \quad \checkmark$$

Corollary: Fourier coeff $a_n, b_n \to 0$ as $n \to \infty$.

Why not obvious? $\quad |a_n| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \right|$

$$\leq \frac{1}{\pi} \cdot \underset{[-\pi,\pi]}{\text{Max}} |f| \cdot \underset{[-\pi,\pi]}{\text{Max}} |\cos nx| \cdot 2\pi$$

$$\leq 2M \cdot 1$$

$$E = \|f\|^2 - 2\sum_{n=1}^{N} A_n c_n + \sum_{n=1}^{N} A_n^2$$

$$E^* = \|f\|^2 - \sum_{n=1}^{N} c_n^2$$

Aha! $E - E^* = \underbrace{\sum_{n=1}^{N} (A_n - c_n)^2}_{\text{sum of squares}} \geq 0$!

$\underline{\underline{E^* \leq E}}$ and equal only if $A_n = c_n$ for each n. ✓

Word: $\{\varphi_n\}$ is a complete orthonormal basis if Bessel's ineq is an equality. (Parseval's Identity).

Fact: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}_{n=1}^{\infty}$ are complete!

Parseval's: $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$

Hilbert space: $L^2[-\pi, \pi]$ = square integrable fcns on $[-\pi, \pi]$.

$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$

$f = \sum_{n=1}^{\infty} c_n \varphi_n$ coordinates!

$L^2[-\pi, \pi] \approx \ell^2$

ℓ^2 = square summable series $\sum c_n^2 < \infty$
$(c_1, c_2, c_3, \ldots)$

Go back and do Bessel's ineq using <u>complex</u> valued φ_n's and <u>complex</u> $\langle \, , \rangle$. Get

$$\sum_{n=1}^{N} |c_n|^2 \leq \int_{-\pi}^{\pi} |f|^2 \, dx$$

$$\| f - \sum^N c_n \varphi_n \|^2 = \int_{-\pi}^{\pi} \left(f - \sum c_n \varphi_n\right) \overline{\left(f - \sum c_n \varphi_n\right)} \, dx$$

$$\bar{f} - \sum \bar{c}_n \bar{\varphi}_n$$

<u>Big fact</u>: $\frac{e^{inx}}{\sqrt{2\pi}}$ are orthonormal

Bessel's $\Rightarrow$ complex $c_n \to 0$ as $n \to \infty$.

$$\langle e^{inx}, e^{imx} \rangle = \int_{-\pi}^{\pi} e^{inx} \, \underbrace{\overline{e^{imx}}}_{e^{-imx}} \, dx$$

$$= \int_{-\pi}^{\pi} e^{inx - imx} \, dx$$

$$= \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx$$

$$\int_{-\pi}^{\pi} \cos(n-m)x \, dx + i \int_{-\pi}^{\pi} \sin(n-m)x \, dx$$

$= 0 + 0i$ if $n \neq m$.

$= 2\pi$ if $n = m$

$$\frac{1}{i(n-m)} e^{i(n-m)} \Big|_{-\pi}^{\pi} \qquad n \neq m$$

$= 0$ (2π- periodic)

$$= \int_{-\pi}^{\pi} 1 \, dx = 2\pi \quad \text{if } n = m.$$