Lesson 7 Fourier series converge! ... but how fast?

Bessel's ineq: φ_n orthonormal, f piecewise continuous on $[-\pi,\pi]$. $c_n = \langle f, \varphi_n \rangle$

$$\sum_{n=1}^{\infty} |c_n|^2 \le \int_{-\pi}^{\pi} |f|^2\,dx. \qquad \text{So } c_n \to 0 \text{ as } n\to\infty.$$

Cor: Fourier coeff $\to 0$ as $n\to\infty$ when f is piecewise continuous.

Today: Assume f is $\mathcal{C}^1$-smooth and 2π-periodic.

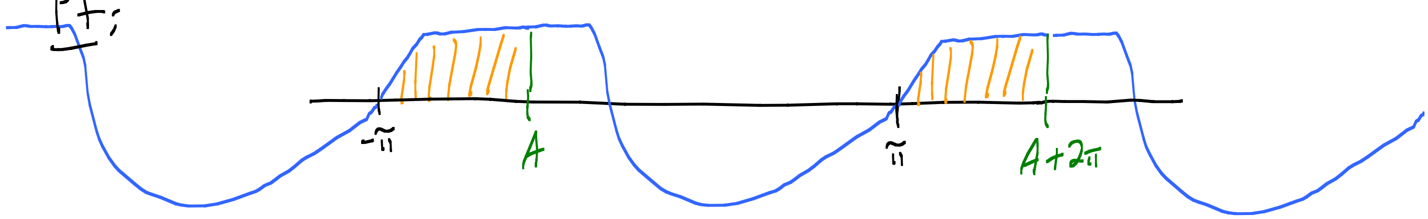
Thm: Fourier series $\displaystyle a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) = \underbrace{\sum_{n=-N}^{N} c_n e^{inx}}_{S_N(x)}$

converges to $f(x)$ at each $x\in[-\pi,\pi]$.

Lemma: f 2π-periodic, then

$$\int_{-\pi}^{\pi} f\,dx = \int_{A}^{A+2\pi} f\,dx \qquad \text{for any } A.$$

Pf:



Pf of Thm: $\displaystyle S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$

$$= \sum_{-N}^{N} \underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\,dt\right)}_{c_n} e^{inx}$$

$$= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\sum_{-N}^{N} e^{in(x-t)}\right) f(t)\,dt$$

Dirichlet Kernel: $$D_N(t) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}$$

Nice: $$S_N(x) = \int_{-\pi}^{\pi} D_N(x-t) f(t)\, dt$$

Properties:

1) $$\int_{-\pi}^{\pi} D_N(t)\, dt = 1$$

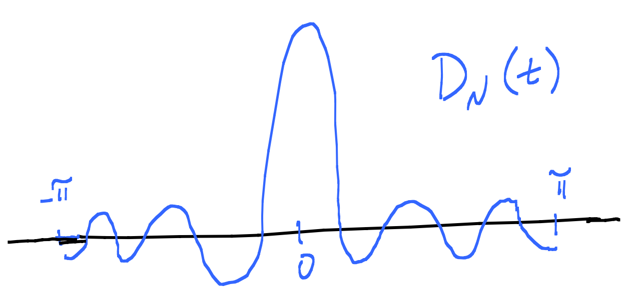
2) $$D_N(t) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}$$

3) L'H shows $$\lim_{t\to 0} D_N(t) = \frac{1}{2\pi} \frac{N+\frac{1}{2}}{\frac{1}{2}} = \frac{2N+1}{2\pi}$$

(Define $D_N(0)$ this way.)

4) $D_N(t)$ is 2π-periodic (because each $e^{\pm inx} = \cos nx \pm i \sin nx$ is)

5)



Bad things about D_N: 1) $D_N(t)$ is not ≥ 0

2) $\displaystyle\int_{-\pi}^{\pi} |D_N(t)|\,dt \longrightarrow \infty$ as $N \to \infty$.

Pf of

$$\int_{-\pi}^{\pi} D_N(t)\,dt = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dt}_{\frac{1}{2\pi}\cdot 2\pi = 1} + \underbrace{\frac{1}{2\pi}\sum_{n\neq 0}^{\pm N}\int_{-\pi}^{\pi} e^{\pm int}\,dt}_{=0}$$

To continue:

$$S_N(x) = \int_{-\pi}^{\pi} D_N(\underbrace{x-t}_{\tau})\, f(t)\,dt$$

$\tau = x - t \Leftarrow t = x - \tau$

$d\tau = -dt$

when $t = -\pi$: $\tau = x + \pi$

$t = \pi$: $\tau = x - \pi$

$$= \int_{\tau = x+\pi}^{x-\pi} D_N(\tau)\, f(x-\tau)\,(-d\tau)$$

$$= \int_{x-\pi}^{x+\pi} D_N(\tau)\, f(x-\tau)\,d\tau$$

2π- Periodic fcn fact (plus change τ to a t) yields

$$S_N(x) = \int_{-\pi}^{\pi} D_N(t)\, f(x-t)\,dt$$

Hmmm. $\displaystyle f(x) = 1 \cdot f(x) = \int_{-\pi}^{\pi} D_N(t)\,dt \cdot f(x)$

$$= \int_{-\pi}^{\pi} D_N(t)\, f(x)\,dt$$

So

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} \underbrace{D_N(t)\left[f(x-t) - f(x)\right]} dt$$

Next

$$\frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}} \left[f(x-t) - f(x)\right]$$

$$= \frac{1}{2\pi} \underbrace{\sin(N+\tfrac{1}{2})t}_{\sin Nt \cos\frac{t}{2} + \cos Nt \sin\frac{t}{2}} \cdot \underbrace{\frac{f(x-t)-f(x)}{\sin\frac{t}{2}}}_{g(t)}$$

L'H $\Rightarrow$ g is nice at $t=0$.

$$S_N(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin Nx \left[\underbrace{\tfrac{1}{2}\cos\tfrac{t}{2} \cdot g(t)}_{G(t)}\right] dt =$$

Fourier sine coeff b_N for G. $\to 0$ as $N \to \infty$

$$+ \frac{1}{\pi}\int_{-\pi} \cos Nx \left[\underbrace{\tfrac{1}{2}\sin\tfrac{t}{2}\, g(t)}_{H(t)}\right] dt =$$

Forier coeff a_N for H $\to 0$ as $N \to \infty$.

Done! But how fast?!

Feeling: The more differentiable f is, the faster the convergence.

Bessel's ineq $\Rightarrow$ Fourier coeff $\to 0$ as $n \to \infty$.

Riemann-Lebesgue lemma: Fourier coeff $\to 0$ as $n \to \infty$

whenever $f \in L^1[-\pi, \pi]$. $\leftarrow$ Lebesgue integrable such that $\underbrace{\int_{-\pi}^{\pi} |f|\, dx}_{L^1\text{-norm of } f.} < \infty$.

Riemann version: f $\underbrace{\text{Riemann-integrable}}_{\text{Bounded and continuous, except maybe on a set of "measure zero"}}$ and $\int_{-\pi}^{\pi} |f|\, dx < \infty$

<u>Riemann-Lebesgue-(your name here) lemma</u>: f C^1-smooth, 2π-periodic, then Fourier coeff $\Rightarrow 0$ as $n \to \infty$.

<u>Pf</u>: $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x)}_{u} \underbrace{\cos nx\, dx}_{dv}$$

$$v = \frac{1}{n} \sin nx$$

$$= \frac{1}{\pi}\left[uv \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} v\, du \right]$$

$$= \frac{1}{\pi}\left[\underbrace{f(x) \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi}}_{=0,\ 2\pi\text{-periodic}} - \int_{-\pi}^{\pi} \frac{1}{n} \sin nx\, f'(x)\, dx \right]$$

So $$|a_n| \le \frac{1}{\pi} \left| \frac{1}{n} \int_{-\pi}^{\pi} \sin nx\, f'(x)\, dx \right|$$

$$\le \frac{1}{n\pi} \int_{-\pi}^{\pi} \underbrace{|\sin nx|}_{\le 1}\, |f'(x)|\, dx$$

$$\leq \frac{1}{n\pi} \underbrace{\max_{[-\pi,\pi]} |f'|}_{=M} \cdot 2\pi$$

$$\leq \frac{1}{n} \cdot 2M$$

Hmmm: f is C^2-smooth. $\underbrace{g(t) = \frac{f(x-t) - f(x)}{\sin \frac{t}{2}}}$

L'H twice shows that $g(t)$ is C^1-smooth.

This means we have shown that Fourier coeff for f C^2-smooth 2π-periodic $\to 0$ as $n \to \infty$ like $\frac{M}{n}$.