Lesson 8 Convergence of Fourier series, Fejér's Theorem

$$S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n x) = \sum_{n=-N}^{N} c_n e^{inx}$$

$c_n = \hat{f}(n)$ Stein

$$S_N(x) = \int_{-\pi}^{\pi} D_N(x-t) f(t)\, dt = \int_{-\pi}^{\pi} D_N(t) f(x-t)\, dt$$

f 2π-periodic (which can be arranged!)

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} \underbrace{D_N(t)}_{\frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}} \left[f(x-t) - f(x)\right] dt$$

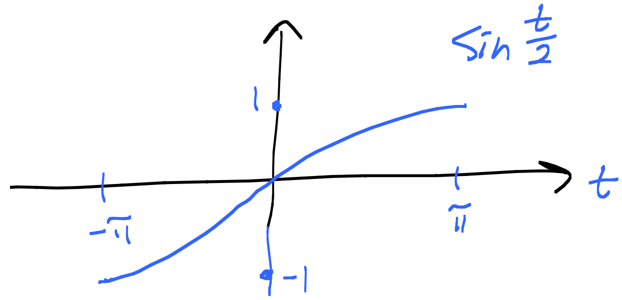
because $\int_{-\pi}^{\pi} D_N\, dt = 1$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(N+\tfrac{1}{2}\right)t \underbrace{\left[\frac{f(x-t) - f(x)}{\sin\frac{t}{2}}\right]}_{g(t)} dt$$

If $g(t)$ is "nice", Bessel's ineq shows this $\to 0$ as $N \to \infty$.

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots = t \underbrace{\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots\right)}_{\text{Radius of conv } = \infty. \text{ (Ratio test)}}$$

$= t\,H(t)$ where H is smooth on $\mathbb{R}$ and $H(0) = 1$.



Suppose g is piecewise continuous (jumps ok) and satisfies a Lipschitz ineq at x:

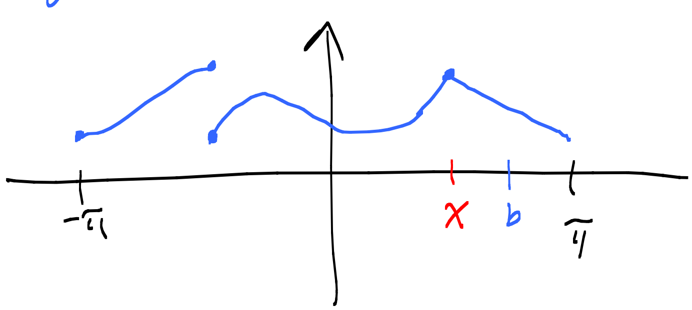
$$|f(x)-f(y)| \le C|x-y|^1$$

$$|f(x-t)-f(x)| \le C|t|^1$$

Aha! Lipschitz estimate shows $g(t)$ stays bounded near 0

Theorem: If g is piecewise continuous (jumps ok) and Lipschitz-1 at x, then $S_N(x) \to f(x)$ as $N \to \infty$.

Note: Suppose g is piecewise C^1-smooth.



g satisfies Lip-1 est at "kinks" via the M.V.T.:

$$\frac{f(x-t)-f(x)}{t} = f'(c) \qquad c \in (x-t, x)$$

So $$\left|\frac{f(x-t)-f(x)}{t}\right| \le \max_{[x,b]}|f'| \quad \checkmark$$

Fourier series converge at smooth places and kinks, but maybe slowly.

Fact: Smooth means faster convergence.

Why: Integration by parts:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$f'(x) \overset{?}{\sim} \sum_{n=-\infty}^{\infty} in\, c_n e^{inx}$$

(yes)

$$f''(x) \overset{?}{\sim} \sum_{n=-\infty}^{\infty} (-n^2) c_n e^{inx}$$

$$\vdots$$

Suppose f is 2π-periodic and as diff'ble as we want

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{u}\, \underbrace{e^{-inx}}_{\cos nx - i\sin nx}\, dx$$

$$dv = e^{-inx}\,dx \qquad u = f(x)$$

$$v = \frac{-1}{in} e^{-inx} \qquad du = f'(x)\,dx$$

$$= \frac{1}{2\pi}\left[\underbrace{\underbrace{uv\Big|_{-\pi}^{\pi}}_{2\pi\text{-periodic!}}}_{=0} - \int_{-\pi}^{\pi} v\, du\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\frac{1}{in}e^{-inx}}_{-v}\ \underbrace{f'(x)\,dx}_{du}$$

$$= \frac{1}{in}\left(\text{Fourier coeff for } f'\right)$$

$$\hat{f}(n) = \frac{1}{in}\ \widehat{f'}(n)$$

$$\boxed{\widehat{f'}(n) = in\,\hat{f}(n)}$$

as if I diff'tiated F.S. term by term.

Cool thing: $\hat{f}(n) = \frac{-1}{n^2}\ \widehat{f''}(n)$

So $$\left|\hat{f}(n)\right| \le \frac{1}{n^2}\left|\ \frac{1}{2\pi}\int_{-\pi}^{\pi} f''(x)\,e^{-inx}\ dx\right|$$

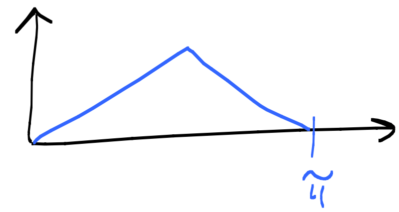
$$\le \frac{1}{n^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f''|\ \underbrace{\left|e^{-inx}\right|}_{=1}\ dx$$

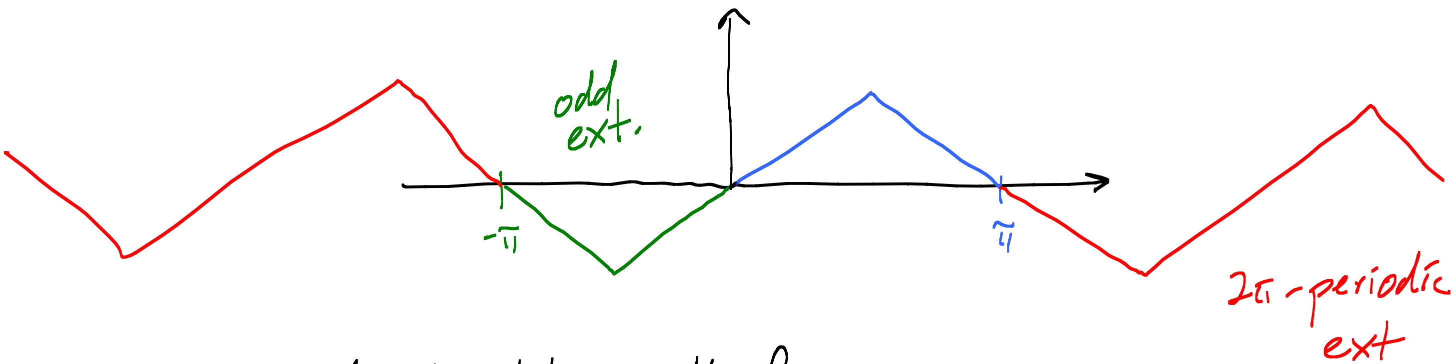
$$\le \frac{1}{n^2}\frac{1}{2\pi}\max_{[-\pi,\pi]}|f''|\cdot 2\pi \le \frac{M}{n^2}$$

Big fact: $\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$ (p-series. $\int$-test)

Conclude via the Weierstraß M-test that Fourier converges uniformly to $f(x)$ when f is 2π-periodic, C^2-smooth.

Think about harpsichord prob:





Get 2π-periodic Lipschitz smooth fcn.

Fourier sine series = Full Fourier series for the extension.

So series converges to f.

Darn! f not C^2-smooth, but...

$$f(x) = c\left(\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right)$$

and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} < \sum_1^\infty \frac{1}{n^2} < \infty$.

M-test ⇒ uniform convergence to f!

Fejér's Theorem: $\underbrace{\frac{S_0(x) + S_1(x) + \cdots + S_{N-1}(x)}{N}}_{\sigma_N(x)}$ ← Cesáro means

converges uniformly to $f(x)$ when f is 2π-periodic and merely continuous!

$$S_n(x) = \int_{-\pi}^{\pi} D_n(t)\, f(x-t)\, dt$$

$$\sigma_N(x) = \int_{-\pi}^{\pi} \underbrace{\frac{D_0(t) + \cdots + D_{N-1}(t)}{N}}_{F_N(t)} f(x-t)\, dt$$

$F_N(t) \leftarrow$ Fejér kernel, Nice!

Facts :

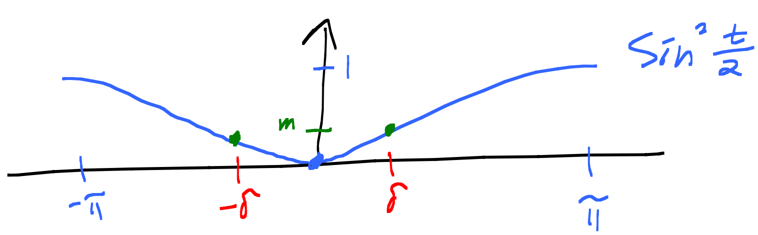
1) $F_N(t) = \dfrac{1}{2\pi N} \dfrac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)} \geq 0$

2) L'H : $\lim\limits_{t \to 0} F_N(t) = \dfrac{1}{2\pi N} \cdot \left[\dfrac{\left(\frac{N}{2}\right)}{\frac{1}{2}}\right]^2 = \dfrac{N}{2\pi}$

$\longrightarrow \infty$ as $N \to \infty$

3) $\displaystyle\int_{-\pi}^{\pi} F_N(t)\, dt = 1$

because $\displaystyle\int_{-\pi}^{\pi} F_N\, dt = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{\int_{-\pi}^{\pi} D_n(t)\, dt}_{=1} = \frac{1}{N} \cdot N = 1$ ✓



4) $\dfrac{1}{2\pi N} \dfrac{\sin^2\left(\frac{Nt}{2}\right)}{\sin^2\left(\frac{t}{2}\right)} \xrightarrow[\text{uniformly!}]{} 0$ on $[-\pi, -\delta] \cup [\delta, \pi]$

5)