Lesson 9 Fejér's Thm and consequences

Fejér's Thm: The averages of the partial sums of the Fourier series

$$\sigma_N(x) = \frac{S_0'(x) + S_1'(x) + \cdots + S_{N-1}'(x)}{N} \qquad S_m'(x) = \sum_{-m}^{m} c_n e^{inx}$$

converge uniformly to $f(x)$ when f is merely continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$. ← f extends 2π-periodically to $\mathbb{R}$ as a continuous fcn.

Pf: Recall $\quad S_N(x) = \int_{-\pi}^{\pi} D_N(t) f(x-t)\, dt$

So

$$\sigma_N(x) = \int_{-\pi}^{\pi} \underbrace{\left[\frac{D_0 + D_1 + \cdots + D_{N-1}(t)}{N}\right]}_{F_N(t),\ \text{the Fejér kernel.}} f(x-t)\, dt$$

Properties:

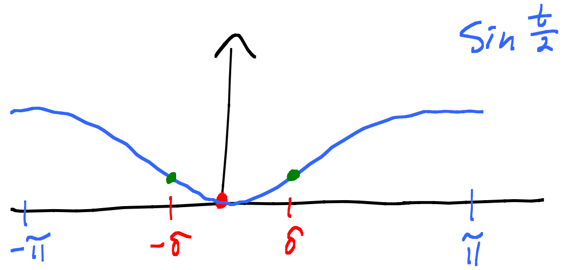
$$F_N(t) = \frac{1}{N} \frac{\sin^2 N\frac{t}{2}}{\sin^2 \frac{t}{2}}$$

1) $F_N(t) \geq 0$ ← formula ✓

2) $\int_{-\pi}^{\pi} F_N(t)\, dt = 1$ ← because D_n do this too

3) $\lim_{t \to 0} F_N(t) = \frac{N}{2\pi} \to \infty$ as $N \to \infty$, BUT

$F_N(t) \to 0$ uniformly as $N \to \infty$ on $[-\pi,-\delta] \cup [\delta, \pi]$ when $0 < \delta < \pi$.



Pf: Fejér kernel is a "good" kernel.

$$\sigma_N(x) - f(x) = \int_{-\pi}^{\pi} F_N(t) f(x-t)\,dt - \underbrace{\int_{-\pi}^{\pi} F_N(t)\,dt}_{=1} \cdot f(x)$$

$$= \int_{-\pi}^{\pi} F_N(t)\,[f(x-t) - f(x)]\,dt$$

$$= \underbrace{\int_{-\delta}^{\delta} F_N(t)\,[f(x-t)-f(x)]\,dt}_{I_1} + \underbrace{\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) F_N(t)[f(x-t)-f(x)]\,dt}_{I_2}$$

Let $\varepsilon > 0$. f continuous $\Rightarrow$ f uniformly continuous on $[-\pi, \pi]$. + 2π-periodic $\Rightarrow$ f unif cont. on $\mathbb{R}$.

So $\exists\, \delta > 0$ such that $|f(x-t) - f(x)| < \frac{\varepsilon}{2}$ when $|t| < \delta$.

$$|I_1| \le \int_{-\delta}^{\delta} \underbrace{|F_N(t)|}_{=F_N(t)}\ \underbrace{|f(x-t) - f(x)|}_{< \frac{\varepsilon}{2}}\,dt$$

$$< \frac{\varepsilon}{2} \underbrace{\int_{-\delta}^{\delta} F_N(t)\,dt}_{\le \underbrace{\int_{-\pi}^{\pi} F_N(t)\,dt}_{=1}} < 1 \cdot \frac{\varepsilon}{2}$$

because $F_N \geq 0$.

$$|I_2| \le \left| \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) F_N(t)\left[f(x-t) - f(x)\right] dt \right|$$

$$\le \underbrace{\left(\max_{\substack{-\pi \le t \le -\delta \\ \delta \le t \le \pi}} |F_N(t)| \right)}_{\text{Can make } < \frac{\varepsilon/2}{4\pi M} \text{ for } N > N_0.} \underbrace{\left(\max_{\mathbb{R}} |f(x-t) - f(x)| \right)}_{\le 2M} \underbrace{\underbrace{\text{Length of intervals}}_{2(\pi-\delta)}}_{< 2\pi}$$

f cont. on $[-\pi, \pi]$. So $|f|$ is cont. It has a max, M.

Get $|I_1 + I_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ for all x when $N > N_0$.

Cor: Trigonometric polys are dense in the space of 2π-periodic continuous fcns, meaning: given $\varepsilon > 0$ and given cont. 2π-periodic fcn f on $\mathbb{R}$, $\exists$

$$T(x) = \sum_{-N}^{N} A_n e^{inx} = \text{sum of } \sin nx,\ \cos nx \text{ terms}$$

such that $|T(x) - f(x)| < \varepsilon$ for all x.

Cor: Weirstraß Thm: Given a continuous fcn f on $[a,b]$, there is a poly $P_N(x) = \sum_{n=0}^{N} a_n x^n$ such

$$|f(x) - P_N(x)| \text{ is as small as desired on } [a,b].$$

Pf: Scale $\mathbb{R}$ so that $[a,b]$ fits in $(-\pi, \pi)$.

Get trig poly close to this.

Finally $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \underbrace{R_n(x)}_{\text{make small on } [-\pi,\pi]}$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots \quad \text{similarly.}$$

Consequence of Fejér's: Bessel's ineq is an equality called Parseval's identity.

Why: $0 \le \underbrace{\Big\| f - \sum_{-N}^{N} \underset{\substack{\uparrow \\ \text{Fourier coeff}}}{c_n} e^{inx} \Big\|^2}_{} \le \underbrace{\Big\| f - \sum_{-N}^{N} \underset{\substack{\uparrow \\ \text{non-Fourier}}}{A_n} e^{inx} \Big\|^2}_{\text{Aha! Can make this small!}}$

$$0 \le \int_{-\pi}^{\pi} |f|^2 dx - \sum_{-N}^{N} |c_n|^2$$

Slow down. Assume f 2π-periodic, continuous.

Get Trig poly $\sum_{-N_0}^{N_0} A_n e^{inx}$ within ε of f on $\mathbb{R}$.

$$\left\| f - \sum_{-N_0}^{N_0} A_n e^{inx} \right\|^2 = \int_{-\pi}^{\pi} \underbrace{\left| f(x) - \sum_{-N_0}^{N_0} A_n e^{inx} \right|^2}_{< \varepsilon^2} dx$$

$$< \varepsilon^2 \cdot 2\pi$$

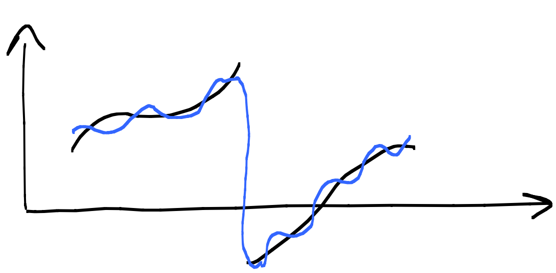
If $N > N_0$. Our trig poly $\sum_{-N_0}^{N_0} A_n e^{inx}$

is included among trig polys of degree N (by setting missing coeff $= 0$).

Get ineq $\quad 0 \le \int_{-\pi}^{\pi} |f|^2 dx - \sum_{-N}^{N} |c_n|^2 \le \varepsilon^2 \cdot 2\pi$

See $\quad \lim_{N \to \infty} \sum_{-N}^{N} |c_n|^2 = \int_{-\pi}^{\pi} |f|^2 dx$

Hmmm. Can make L^2 norm small even when Max big!



Parseval's will hold for fcns with jumps!