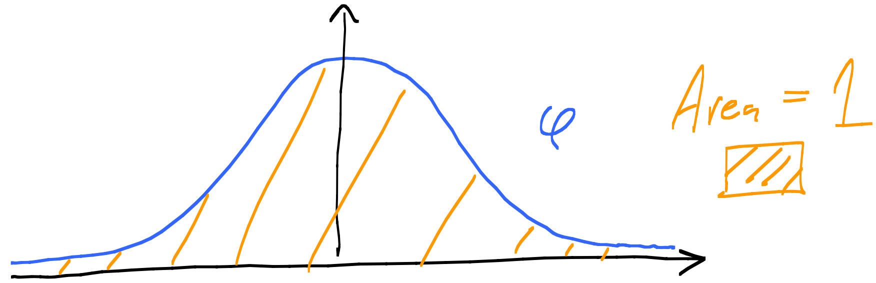
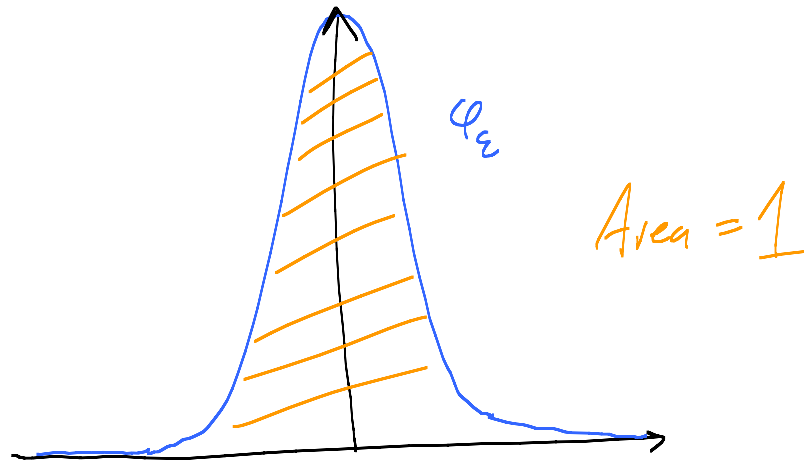
Lesson 11 "Good" kernels



<u>Ex</u>: $\varphi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$



$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$$

$u = \frac{x}{\varepsilon}$

<u>Famous calculation</u>: $$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$

$$= \int_0^\infty \int_0^\infty \underbrace{e^{-x^2} e^{-y^2}}_{e^{-(x^2+y^2)}} \underbrace{dx\,dy}_{r\,dr\,d\theta} \quad \leftarrow \text{first quadrant}$$

Aha! $$= \int_0^{\pi/2} \frac{1}{2}\int_0^\infty e^{-r^2}\, 2r\,dr\,d\theta \qquad r^2 = u, \quad 2r\,dr = du$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[-e^{-u}\right]_0^\infty d\theta = \frac{\pi}{4}$$

$$\left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)^2 = \left(2\int_0^\infty\right)^2 = 4\cdot\frac{\pi}{4} = \pi \quad \checkmark$$

Famous use: $\displaystyle\int_{-\pi}^{\pi} F_N(t)\, f(x-t)\, dt \leftarrow$ convolution

On $\mathbb{R}$: $$\varphi_\varepsilon * f = \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, f(x-t)\, dt$$

Fact: $\varphi_\varepsilon * f = f * \varphi_\varepsilon$

Feeling: $\varphi_\varepsilon * f$ smooths out f and approximates f as $\varepsilon \to 0$.

Why: $$\varphi_\varepsilon * f = (f * \varphi_\varepsilon)(x) = \int_{-\infty}^{\infty} f(t)\, \varphi_\varepsilon(x-t)\, dt$$

$$DQ = \frac{(f * \varphi_\varepsilon)(x+h) - (f * \varphi_\varepsilon)(x)}{h} = \int_{-\infty}^{\infty} f(t) \left(DQ \text{ on } \varphi_\varepsilon ! \right) dt$$

If f is continuous and bounded (or even of "exponential type", meaning that $|f(x)| \le K e^{c|x|}$ on $\mathbb{R}$), see that

$\varphi_\varepsilon * f$ is ∞-diff'ble.

Claim: $\varphi_\varepsilon * f \to f$

Why: $$(\varphi_\varepsilon * f)(x) - f(x) \cdot \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, dt}_{=1}$$

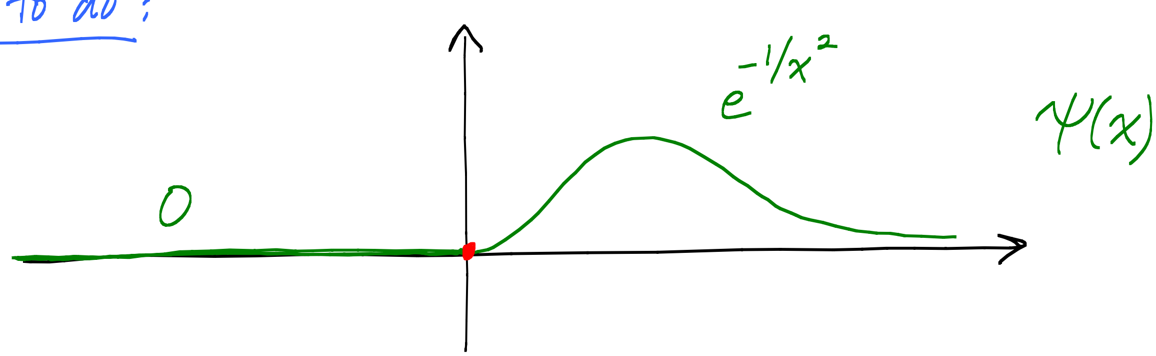
$$= \int_{-\infty}^{\infty} \left[\varphi_\varepsilon(t)\, f(x-t) - \varphi_\varepsilon(t)\, f(x)\right] dt$$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(t) \left[f(x-t) - f(x) \right] dt$$

$$= \underbrace{\int_{-\delta}^{\delta} \varphi_\varepsilon(t) \underbrace{\left[f(x-t) - f(x) \right]}_{\substack{\text{small} \\ |\quad| < \varepsilon}} dt}_{< \int_{-\delta}^{\delta} \underbrace{|\varphi_\varepsilon|}_{=\varphi_\varepsilon} \cdot \varepsilon \; dt \;<\; \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon \, dt}_{=1!} \cdot \varepsilon} + \underbrace{\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \underbrace{\varphi_\varepsilon(t)}_{\text{small}} \underbrace{\left[f(x-t) - f(x) \right]}_{\text{bounded}} dt}_{2M \underbrace{\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \varphi_\varepsilon \, dt}_{\text{Yes, this does } \to 0 \text{ as } \varepsilon \to 0.}}$$

This how analysts show C^∞ functions are "dense" among continuous fcns.

Fun thing to do:



$$\Psi(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x^2} & x > 0 \end{cases}$$

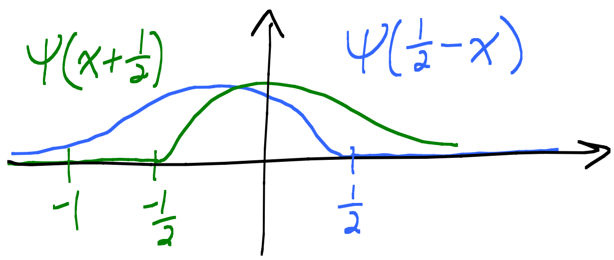
Clearly continuous at 0.

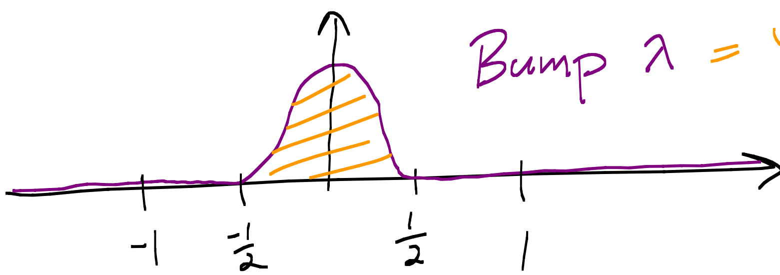
$$\text{DQ at } 0 = \frac{\psi(0+h)-\psi(0)}{h} = \frac{e^{-1/h^2}-0}{h} \xrightarrow{\text{L'H}} 0 \quad \text{as } h \to 0.$$

Repeat! See ψ is C^∞ smooth. All deriv $\psi^{(n)}(0)$ are zero!

Classic example of C^∞ fcn not equal to its Taylor series at origin.

Trick:





Bump $\lambda = \psi(x+\frac{1}{2})\cdot\psi(\frac{1}{2}-x)$

Let $c = \int_{-1/2}^{1/2} \lambda(x)\,dx > 0.$

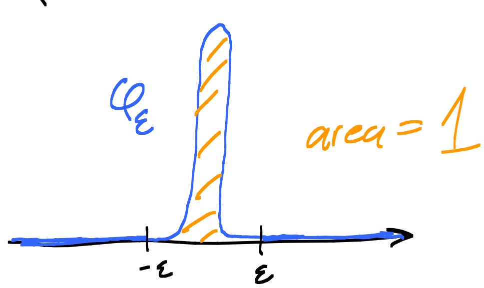
Finally $\varphi(x) = \frac{1}{c}\lambda(x).$

$$\boxed{\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)}$$ ← Really good kernel

1) Support of $\varphi_\varepsilon \subset (-\varepsilon, \varepsilon)$. $\left(\text{Supp } \varphi_\varepsilon = \overline{\{x : \varphi_\varepsilon(x) \neq 0\}}\right)$

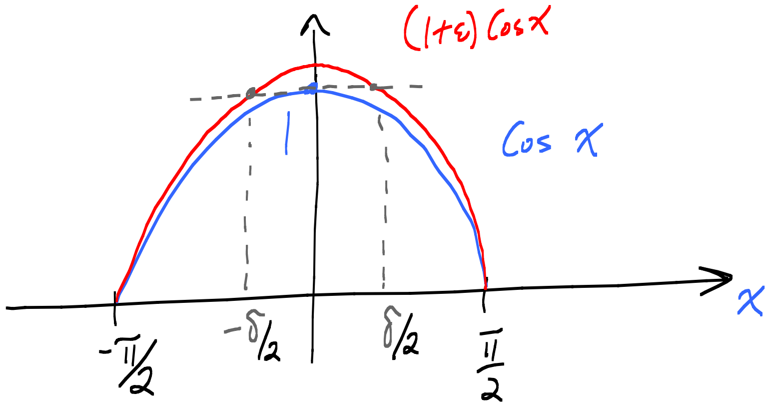
2) $\varphi_\varepsilon(x) \geq 0$

3) $\int_{-\infty}^{\infty} \varphi_\varepsilon(x)\,dx = 1$



4) Given $\delta > 0$, $\varphi_\varepsilon(x) \to 0$ outside $(-\delta, \delta)$ as $\varepsilon \to 0$.

Stein:



Take powers. Make it small as you want outside $(-\delta, \delta)$.

Make $\int_{-\pi/2}^{\pi/2} [(1+\varepsilon)\cos x]^N \, dx$ get as big as desired.

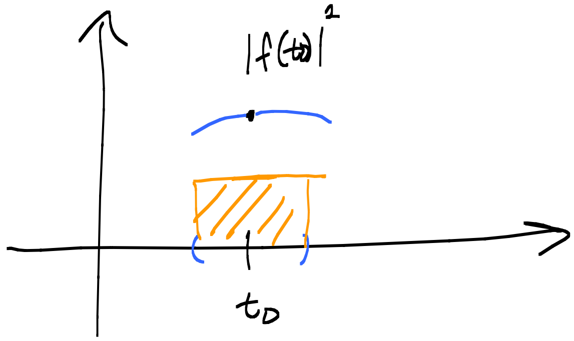
Can divide by $c = \int_{-\pi/2}^{\pi/2}$ without messing up smallness part to make $\int = 1$.

Stein uses it to show that

Thm: If f is piecewise continuous and $\hat{f}(n) = 0$ for all n, then f must be zero where it is continuous.

Our pf: Parseval's shows

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 \, dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = 0$$



$$\varepsilon = \frac{|f(t_0)|^2}{2}$$

$$\delta > 0$$

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt > 2\delta \cdot \varepsilon > 0 \quad \lightning$$