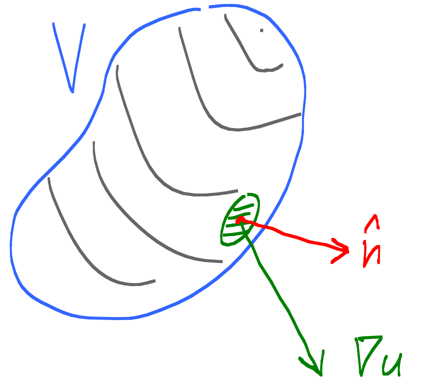
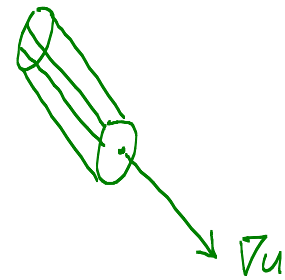
Lesson 14 The Poisson formula for the solution to the Dirichlet problem

Exam 1 in class, Wed., March 4

Control Volume V



u = Temp in kelvin



$$\frac{d}{dt}\Big(\underbrace{k\iiint_V u \, dx\,dy\,dz}_{\text{heat content of } V}\Big) = \underbrace{c\iint_{\partial V} \nabla u \cdot \hat{n} \, \underbrace{d\sigma}_{\text{surface area}}}_{\text{use the divergence theorem (Gauss' Thm).}}$$

$$\iint_{\partial V} \vec{F}\cdot\hat{n}\,d\sigma = \iiint_V \operatorname{div}\vec{F}\,d\text{vol}$$

$$k\iiint_V \frac{\partial u}{\partial t}\,d\text{vol} = c\iiint_V \underbrace{\operatorname{div}\nabla u}_{\Delta u}\,d\text{vol}$$

$$\operatorname{div}(F_1\hat{\imath}+F_2\hat{\jmath}+F_3\hat{k}) = \frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}+\frac{\partial F_3}{\partial x_3}$$

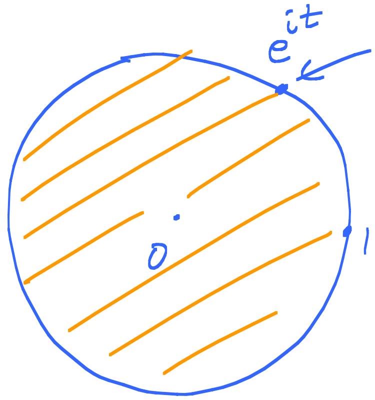
$$\iiint_V \Big(\underbrace{k\frac{\partial u}{\partial t} - c\,\Delta u}\Big)\,d\text{vol} = 0$$

Aha! If this is $\neq 0$ at p_0, take a very small $V = \text{Ball}_\varepsilon(p_0)$ to make $\iiint$ also $\neq 0$. $\lightning$

Conclude the heat eqn: $\frac{\partial u}{\partial t} = \left(\frac{c}{k}\right)\Delta u.$ ✓

Steady state: $0 = \Delta u.$

Dirichlet prob



Given $f(t)$ on boundary, find u continuous on $\overline{D_1(0)}$, harmonic inside, and

$$u(\underset{r}{\underset{\uparrow}{1}}, \underset{\theta}{\underset{\uparrow}{t}}) = f(t)$$

Got

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \big(\underbrace{a_n \cos n\theta + b_n \sin n\theta}\big)$$

$$\underbrace{a_n \frac{e^{in\theta} + e^{-in\theta}}{2} + b_n \frac{e^{in\theta} - e^{-in\theta}}{2i}}$$

$$\underbrace{\frac{(a_n - i b_n)}{2}}_{c_n} e^{in\theta} + \underbrace{\frac{(a_n + i b_n)}{2}}_{c_{-n}} e^{-in\theta}$$

a's, b's = Fourier series for f

$$u(r,\theta) = \overset{c_0}{\overset{\|}{a_0}} + \sum_{n=1}^{\infty} \left(c_n r^n e^{in\theta} + c_{-n} r^n e^{-in\theta} \right)$$

$$= \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

$$e^{in\theta} \cdot e^{-int} = e^{in(\theta - t)}$$

Partial sum $$\sum_{n=-N}^{N} r^{|n|} \Big(\underbrace{\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt}_{c_n} \Big) e^{in\theta}$$

$$= \int_0^{2\pi} \Big(\underbrace{\frac{1}{2\pi} \sum_{n=-N}^{N} r^{|n|} e^{in(\theta - t)}} \Big) f(t)\, dt$$

$$w = e^{i(\theta - t)}$$
$$\overline{w} = e^{-i(\theta - t)}$$

$$\sum_{n=-N}^{N} = \sum_{n=0}^{N} + \sum_{n=-N}^{-1}$$

$$= \left(1 + rw + r^2 w^2 + \cdots + r^N w^N\right) + \left(r\bar{w} + r^2\bar{w}^2 + \cdots + r^N \bar{w}^N\right)$$

$$= \underbrace{\left(1 + (rw) + (rw)^2 + \cdots + (rw)^N\right)}_{\frac{1-(rw)^{N+1}}{1-rw}} + (r\bar{w})\left(1 + (r\bar{w}) + (r\bar{w})^2 + \cdots + (r\bar{w})^{N-1}\right)$$

Geom series!

$$\frac{1-(rw)^{N+1}}{1-rw} + (r\bar{w})\frac{1-(r\bar{w})^N}{1-r\bar{w}}$$

Let $N \to \infty$. $\left|(rw)^m\right| = r^m \underbrace{\left|w^m\right|}_{\underbrace{\left|e^{i(\theta-t)}\right|^m}_{=1}} = r^m \to 0$ as $m \to \infty$.

$$\sum_{-N}^{N} \to \frac{1}{1-rw} + \frac{r\bar{w}}{1-r\bar{w}} = \frac{1 - r\bar{w} + r\bar{w} - r^2 \overset{=1}{w\bar{w}}}{\underbrace{(1-rw)(1-r\bar{w})}_{|1-rw|^2}}$$

$$= \frac{1-r^2}{|1-rw|^2}$$

$|zw| = |z|\,|w|$

$$= \frac{1-r^2}{\left|1-re^{i(\theta-t)}\right|^2} \cdot \underbrace{\frac{1}{\left|e^{it}\right|^2}}_{1}$$

$$= \frac{1-r^2}{\left|e^{it}-re^{i\theta}\right|^2}$$

Poisson formula:

$$u(r,\theta) = \int_0^{2\pi} \underbrace{\frac{1}{2\pi} \frac{1-r^2}{|re^{i\theta} - e^{it}|^2}}_{P(r,\theta,t)} f(t)\, dt$$

$P(r,\theta,t) \leftarrow$ Poisson kernel

Key properties:

1) $P(r,\theta,t) > 0$ formula ✓

2) $\int_0^{2\pi} P(r,\theta,t)\, dt = 1$ for $0 \le r < 1$.

$$\int_0^{2\pi} P(r,\theta,t)\, dt = \lim_{N\to\infty} \underbrace{\int \sum_{-N}^{N}}_{\sum_{-N}^{N} \frac{1}{2\pi}\int_0^{2\pi} r^{|n|} \underbrace{e^{in(\theta-t)}}\, d\theta}$$

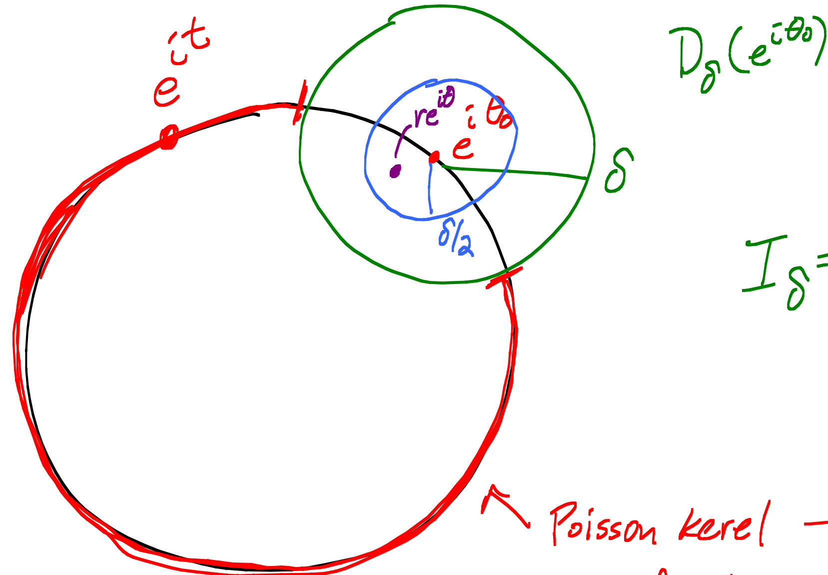
integrate to zero, $n \neq 0$

$$= r^0 \cdot e^0 = 1$$

4) $\Delta_{(r,\theta)} P(r,\theta,t) \equiv 0.$ formula ✓

Easy in form $\dfrac{1-r^2}{1+r^2-2r\cos(\theta-t)}$

3)



$$I_\delta = \{t : e^{it} \notin D_\delta(e^{i\theta_0})\}$$

Poisson kerel $\to 0$ unif here.

Claim $P(r,\theta,t) \to 0$ uniformly for $t \in I_\delta$ when

$re^{i\theta} \in D_{\delta/2}(e^{i\theta_0})$, $r \to 1$.

$$P(r,\theta,t) = \frac{1}{2\pi}\frac{1-r^2}{|re^{i\theta}-e^{it}|^2} \le \frac{1}{2\pi}\frac{(1-r)\overbrace{(1+r)}^{<2}}{\left(\frac{\delta}{2}\right)^2} < \frac{1-r}{\pi \cdot \frac{\delta^2}{4}}$$

$\to 0$ unif ✓