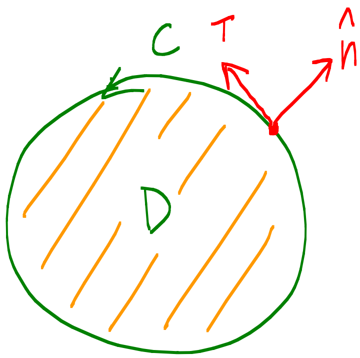
Lesson 17 Green's identities and the Dirichlet problem

HWK 5 due next Fri
Exm 1 in class, Wed, March 4



Green's Theorem:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Divergence Theorem (Gauss' Thm)

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_D \underbrace{\mathrm{Div}\,\vec{F}}_{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}}\,dx\,dy$$

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath}$$

$$\mathrm{Div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

Aha! Take $F_1 = Q$
$F_2 = -P$

$$= \int_C P\,dx + Q\,dy$$

$$= \int_C -F_2\,dx + F_1\,dy$$

$C: \ (x(t), y(t)), \ 0 \le t \le 2\pi$

Let t = arc length

$$T = \underbrace{\dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}}_{\text{unit vector if } t = \text{arc length}}$$

$$= \int_0^{2\pi} -F_2(x(t),y(t))\,\dot{x}(t)\,dt + \int_0^{2\pi} F_1(x(t),y(t))\,\dot{y}(t)\,dt$$

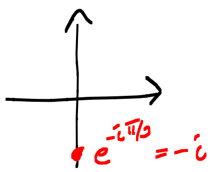
$$= \int_0^{2\pi} (F_1\hat{\imath} + F_2\hat{\jmath})\cdot\underbrace{(\dot{y}(t)\hat{\imath} - \dot{x}(t)\hat{\jmath})}\,dt$$

$\perp$ to T
unit vector too!
So its $\pm\hat{n}$

Claim: It's the <u>outward</u> pointing normal.

Represent T by a complex number: $\dot{x} + i\dot{y}$

Representation of $\hat{n}$ by a complex number is



$$e^{-i\frac{\pi}{2}}\, T = \hat{n}$$

$$-i\,(\dot{x} + i\dot{y})$$

$$= \dot{y} - i\dot{x}$$

Done!

$\dot{y}\,\hat{i} - \dot{x}\,\hat{j}$ is the outward pointing unit normal vector $\hat{n}$.

$$e^{i\alpha}\cdot(\underbrace{z}_{re^{i\theta}}) = re^{i(\theta+\alpha)}$$

$+\alpha$: counterclockwise rotation by α

$$e^{-i\alpha}\cdot z = re^{i(\theta-\alpha)}$$

$-\alpha$: clockwise rotation,

Green's 1st identity: $\vec{F} = u\nabla v$

Div Thm: $$\int_C \underbrace{\vec{F}\cdot\hat{n}}_{u\underbrace{\nabla v\cdot\hat{n}}_{\frac{\partial v}{\partial n}}}\, ds = \iint_D \underbrace{\mathrm{Div}(u\nabla v)}_{}\, dx\,dy$$

$$\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}\right)$$

$$\nabla u\cdot\nabla v + u\,\Delta v$$

$$\int_C u\frac{\partial v}{\partial n}\, ds = \iint_D \nabla u\cdot\nabla v + u\,\Delta v\; dx\,dy$$

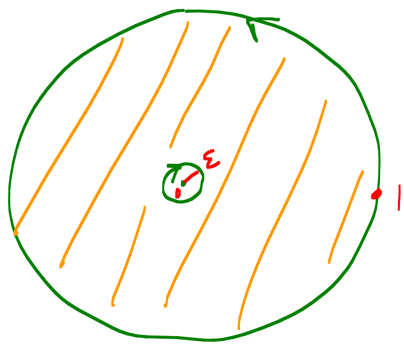
Green's 2nd:

$$\int_C v\frac{\partial u}{\partial n}\, ds = \iint_D \nabla v\cdot\nabla u + v\,\Delta u\; dx\,dy$$

Subtract: $\displaystyle\int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = \iint_D u\Delta v - v\Delta u \, dx\,dy$ ✓

u = harmonic $\quad v = \ln r$ harmonic too.

Green's 2nd



Get averaging property for harmonic fcns!

Lazy person's approach to the Dirichlet problem

Notice that $r^n \sin n\theta$, $r^n \cos n\theta$, 1 are all harmonic.

Hmmm: When $r=1$, these are the generators of all trig polys! Given continuous real valued 2π-periodic fcn $f(\theta)$ and $\varepsilon > 0$, Fejér's Thm $\Rightarrow$

Trig poly $\displaystyle P_N(\theta) = A_0 + \sum_{n=1}^{N} \left(A_n \cos n\theta + B_n \sin n\theta\right)$

such that $|P_N(\theta) - f(\theta)| < \varepsilon$ on $[0, 2\pi]$.

Aha! $\displaystyle u_N(r,\theta) = A_0 + \sum_{n=1}^{N} \left(A_n r^n \cos n\theta + B_n r^n \sin n\theta\right)$

is <u>harmonic</u> and $u_N(1,\theta) = P_N(\theta) \sim f(\theta)$

Lazy idea: Let $\varepsilon = \frac{1}{n}$. Get a u_n with its P_n

such that $\left| \underbrace{P_n(\theta)}_{u_n(1,\theta)} - f(\theta) \right| < \frac{1}{n}$

<u>Big question</u>: $P_n \to f$ uniformly.

Do $u_n(r,\theta)$ converge to sol^n to D. Prob?

Yes, they do!

Tools for lazyness: Ave prop., max princ.

<u>Facts</u>: $P_n \to f$ uniformly $\Rightarrow$ P_n uniformly Cauchy

meaning, given $\varepsilon > 0$, there is an N such that

$|P_n(\theta) - P_m(\theta)| < \varepsilon$ for <u>all</u> θ when $n, m > N$.

uniformly

Know max princ for harmonic fcns.

$$|u_n(r,\theta) - u_m(r,\theta)| \leq \max_{\theta} \left| \underbrace{u_n(1,\theta)}_{P_n(\theta)} - \underbrace{u_m(1,\theta)}_{P_m(\theta)} \right|$$

So $u_n(r,\theta)$ is uniformly Cauchy on $\overline{D_1(0)}$.

Fact: A unif Cauchy seq of cont fcns on a closed and bounded set converges uniformly.

Fact: Uniform limits of continuous fcns are cont.

MA 341. Get $u(r,\theta) = \lim_{n\to\infty} u_n(r,\theta)$ continuous on $\overline{D_1(0)}$ and $u(1,\theta) = f(\theta)$.

Last big question: Is u harmonic?

Sneaky plan:

$$\text{Ave of } u \text{ on } C_r(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \overset{\text{want}}{=} u(z_0)$$

$$\text{Ave of } u_n \text{ on } C_r(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u_n(z_0 + re^{i\theta})\,d\theta = u_n(z_0)$$

$$\left| \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta - u(z_0) \right|$$

$$\left| \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0+re^{i\theta}) - u_n(z_0+re^{i\theta})}_{\text{close together}}\,d\theta + \underbrace{\int_0^{2\pi} u_n(z_0+re^{i\theta})\,d\theta - u_n(z_0)}_{=0} + \underbrace{u_n(z_0) - u(z_0)}_{\to 0} \right|$$

See $|\sim|$ is arbitrarily small.

Conclude: u satisfies ave prop! So it's harmonic.

www.purdue.edu/~bell/papers ← first one

Poisson / Dirichlet
the easy way!