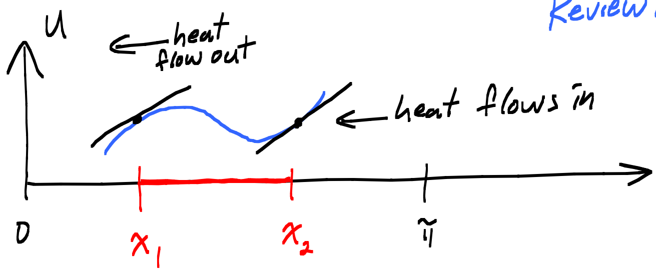
Lesson 19 Hot wire problems

Hwk 5 due Friday, Exam 1 in class Wed

Review material for exam on home page



$$\frac{\partial}{\partial t}\left(\text{Heat content in } [x_1,x_2]\right) = \begin{pmatrix}\text{rate of heat}\\ \text{flow } \underline{\text{in}}\\ \text{from right}\end{pmatrix} + \begin{pmatrix}\text{rate of heat}\\ \text{flow } \underline{\text{in}}\\ \text{from left}\end{pmatrix}$$

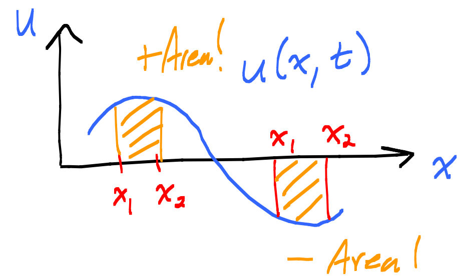
$$\frac{\partial}{\partial t}\left(c\int_{x_1}^{x_2} u(x,t)\,dx\right) = k\frac{\partial u}{\partial x}(x_2,t) - k\frac{\partial u}{\partial x}(x_1,t)$$

$$\int_{x_1}^{x_2} c\frac{\partial u}{\partial t}(x,t)\,dx = k\int_{x_1}^{x_2}\frac{\partial^2 u}{\partial x^2}(x,t)\,dx$$

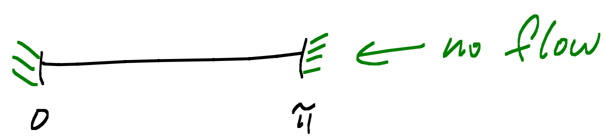
Fund. Thm. Calc.

$$0 = \int_{x_1}^{x_2}\underbrace{\left(c\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}\right)}_{\text{must be } \equiv 0}(x,t)\,dx$$

must be $\equiv 0$ because $[x_1,x_2]$ is arbitrary!



Hot wire with insulated ends:



But: Kelvin's says (rate of heat flow) prop $-\underbrace{(\text{temp gradient})}_{\text{must be zero!}}$

Boundary conditions: $\begin{cases}\dfrac{\partial u}{\partial x}(0,t) = 0 & \text{for all } t\\[2ex] \dfrac{\partial u}{\partial x}(\pi,t) = 0 & \text{for } \forall t\end{cases}$

Separation of vars: $u(x,t) = X(x)T(t)$

BC's: $\dfrac{\partial u}{\partial x}(0,t) = X'(0)T(t) \overset{\text{want}}{=} 0$

don't want $T(t) \equiv 0$.

Need $X'(0)=0$. Similarly, need $X'(\pi)=0$ too.

Do the routine:

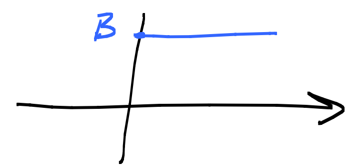
$$X''(x) - \lambda X(x) = 0 \qquad\qquad T'(t) - \lambda T(t) = 0$$

$$\text{BC} \begin{cases} X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

Case $\lambda = 0$: $X(x) = Ax + B$

$$X'(x) \equiv A$$

$$X'(0) = A = X'(\pi) \leftarrow \text{need } A=0$$



Get a non-zero sol^n $X_0(x) \equiv 1$.

T prob: $T' \equiv 0$. $T(t) =$ const. Take $T_0(t) \equiv 1$.

$$\boxed{u_0(x,t) = 1}$$

Case $\lambda > 0$: $\lambda = k^2$ $\quad X'' - \lambda X = 0$

$$r^2 - k^2 = 0$$

$$r = \pm k$$

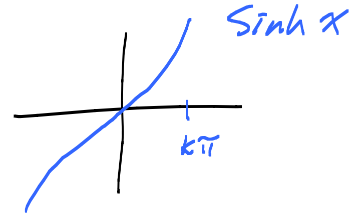
$$X(x) = A\cosh kx + \cancel{B\sinh kx} \leftarrow \text{or } ae^{kx} + be^{-kx}$$

$$X'(x) = Ak\sinh kx + Bk\cosh kx$$

$$X'(0) = Ak\cdot 0 + Bk\cdot 1 \overset{\text{want}}{=} 0 \qquad \boxed{B=0}$$

$$X'(\pi) = A\,k\,\underbrace{\sinh k\pi}_{\text{not zero}} \overset{\text{want}}{=} 0$$

A $\uparrow$ $\neq 0$



Must have $\boxed{A=0}$. Only get zero sol^n!

Case $\lambda<0$: $\lambda=-k^2$

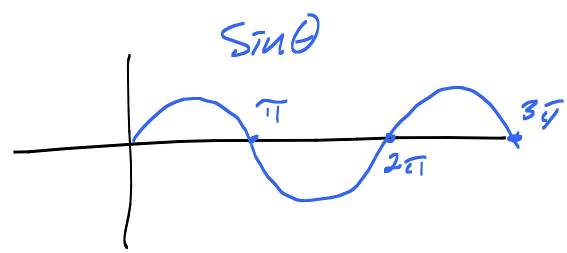
$$X(x) = A\cos kx + \cancel{B\sin kx}$$

$$X'(x) = -kA\sin kx + kB\cos kx$$

$$X'(0) = -kA\cdot 0 + kB\cdot 1 \overset{\text{want}}{=} 0 \qquad \boxed{B=0}$$

$$X'(\pi) = -kA\,\underbrace{\sin k\pi}_{\text{need}=0.} \overset{\text{want}}{=} 0$$

don't want $A=0$



Need $k\pi = n\pi$, $n=1,2,3,\ldots$

$\boxed{k=n}$ $\qquad \lambda=-k^2=-n^2$

$$X_n(x) = A\cos nx \qquad \text{Take } A=1$$

$$T' - (-n^2)T = 0 \qquad T' = -n^2 T \quad \text{exponential decay}$$

$$T(t) = Ce^{-n^2 t}$$

Take $C=1$ now

$$X_n(x) = \cos nx$$

$$T_n(t) = e^{-n^2 t}$$

$$\boxed{u_n(x,t) = e^{-n^2 t} \cos nx}$$

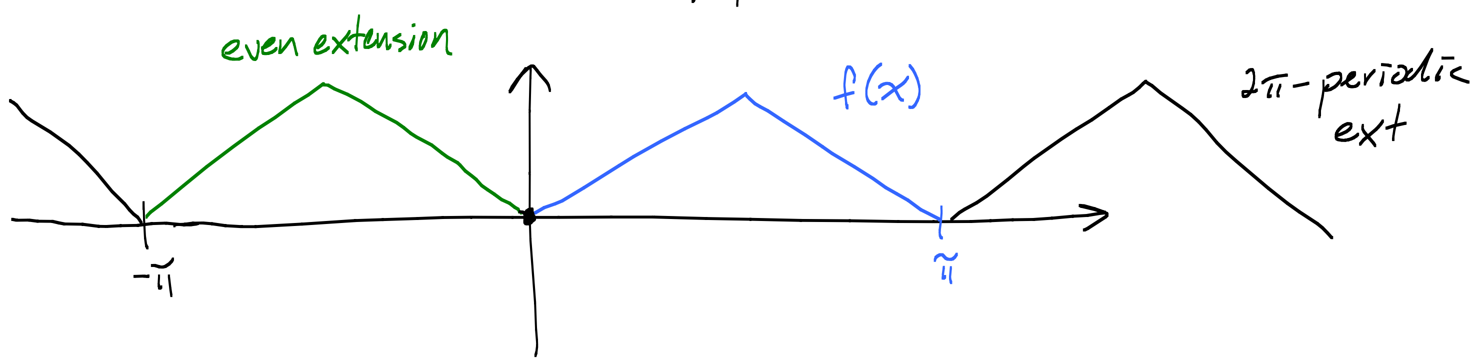
PDE is <u>linear</u> and BC are "homogeneous". Can take linear combos!

Try $u(x,t) = a_0 u_0(x,t) + \sum_{n=1}^{\infty} a_n u_n(x,t)$

$$\boxed{u(x,y) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx}$$

<u>Initial condition</u>: $u(x,0) = f(x)$ ← Known

Need $u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ ← Fourier cosine series!



$F(x)$ = 2π-periodic <u>even</u> extension of f has <u>no</u> sine terms.

$\underbrace{\text{Fourier series for } F}_{\text{Fourier cosine series series for } f}$ on $[-\pi,\pi]$ converges to F (no jumps!) converges to f on $[0,\pi]$.

Fascinating thing:

$$\lim_{t\to\infty}\left[a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos nx\right] = a_0$$

die out very fast as $t\to\infty$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} f(x)\,dx = \left(\begin{array}{c}\text{Ave of } f\\ \text{on } [0,\pi]\end{array}\right)$$

Homework problem: $X'' - \lambda X = 0$

$X'(0) = 0 \leftarrow$ insulate $x=0$ end

$X(\pi) = 0 \leftarrow$ refrigerate $x=\pi$ end (hold it to zero degrees)

Get $X_n(x)$ for a seq. of λ's.

Sturm-Liouville $\Rightarrow$ X_n's are $\perp$ on $[0,\pi]$!

Generalized Fourier series:

$$f \overset{\text{want}}{=} \sum_{n=1}^{\infty} c_n X_n$$

$$\int_0^{\pi} X_m \cdot f\,dx = \int_0^{\pi}\left(\sum_{n=1}^{\infty} c_n X_n\right)\cdot X_m\,dx$$

$$= \sum_{n=1}^{\infty} c_n \underbrace{\int_0^{\pi} X_n(x) X_m(x)\, dx}_{=0 \text{ when } n \neq m}$$

$$= c_m \int_0^{\pi} X_m(x)^2\, dx$$

See $$c_m = \frac{\int_0^{\pi} f(x) X_m(x)\, dx}{\int_0^{\pi} X_m(x)^2\, dx}$$ ← c_m determined

Cool thing: X_n's are like eigenvectors!

$$L = \frac{d^2}{dx^2}$$

$$L X_n = \lambda X_n$$

eigenvalue!