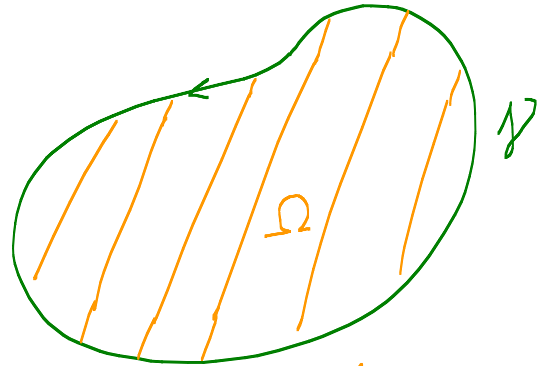
Lesson 21 The Isoparametric inequality via Fourier series!



Isoparametric inequality:

$$A \le \frac{L^2}{4\pi}$$

equality only if Ω is a disc.

A = area

L = Length of γ

Circle case: $A = \pi r^2 \leftarrow \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$

$L = 2\pi r \leftarrow r = \frac{L}{2\pi}$

Key ingredients: 1) Green's theorem:

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\substack{Q = x \quad P = -y \\ =2}} dx\,dy$$

$$\int_\gamma -y\,dx + x\,dy = 2\,\text{Area}(\Omega)$$

Big fact: $\text{Area}(\Omega) = \frac{1}{2}\underbrace{\int_a^b -y(t)\,\dot{x}(t) + x(t)\,\dot{y}(t)\,dt}_{\text{inner products!}}$

Key #2: $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$F(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

Parseval's $\Longrightarrow$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,F(x)\,dx = 2a_0A_0 + \sum_{n=1}^{\infty}\left(a_nA_n + b_nB_n\right)$$

<u>Philosophy</u>: Parsevals $\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}|f|^2\,dx}_{L^2\text{-norm squared}} = \underbrace{2a_0^2 + \sum\left(a_n^2 + b_n^2\right)}_{\ell^2\text{-norm squared}}$

Fact in Hilbert space theory: Norms preserved $\Longrightarrow$ $\langle\ ,\ \rangle$ preserved

<u>Why</u>: Parseval's for $f+F$ and $f-F$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|f+F|^2}_{f^2+2fF+F^2}\,dx = 2(a_0+A_0)^2 + \sum_{n=1}^{\infty}\left[(a_n+A_n)^2 + (b_n+B_n)^2\right]$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|f-F|^2}_{f^2-2fF+F^2}\,dx = 2(a_0-A_0)^2 + \sum_{n=1}^{\infty}\left[(a_n-A_n)^2 + (b_n-B_n)^2\right]$$

Subtract!

$$\frac{4}{\pi}\int_{-\pi}^{\pi} f(x)F(x)\,dx = 4\left[2a_0A_0 + \sum_{n=1}^{\infty}\left(a_nA_n + b_nB_n\right)\right] \checkmark$$

Fact: $c_n = \dfrac{a_n - ib_n}{2}$, $\quad c_{-n} = \dfrac{a_n + ib_n}{2}$ $\qquad n > 0$

So $a_n = c_n + c_{-n}$ $\qquad c_{-n} - c_n = i\, b_n$

$b_n = i\,(c_n - c_{-n})$

Plug these in above. Get

Parseval's #2:
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{F(x)}\,dx = \sum_{n=-\infty}^{\infty} c_n \overline{C_n}$$

Steps to simplify calculations:

1) Rescale so that $L = 2\pi$.

2) Parametrize γ: $z(t)$, $-\pi \le t \le \pi$

via arc length t.

3) Assume γ is C^2-smooth so that $z(t) = x(t) + i\,y(t)$

are C^2-smooth.

Want to see $A \le \dfrac{L^2}{4\pi} = \dfrac{(2\pi)^2}{4\pi} \le \pi$

Notation: $x(t)$ and $y(t)$ from parametrization wrt arc length t.

$\hat{x}(n) = c_n$ for $x(t)$ (complex F. coeff).

$\hat{y}(n) = \cdots\cdots$ $y(t)$

We know $\hat{\dot{x}}(n) = in\,\hat{x}(n)$

$\hat{\dot{y}}(n) = in\,\hat{y}(n)$

Guess this by diff'tiating $x(t) = \sum_{-\infty}^{\infty} \hat{x}(n)\, e^{int}$

Prove it via $\hat{\dot{x}}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \dot{x}(t)\, e^{-int}\, dt$

$u = e^{-int}$

$dv = \dot{x}(t)\,dt$

... integrate by parts ...

$du = -in\,e^{-int}\,dt$

$v = x(t)$

$= in\,\hat{x}(n)$ ✓

<u>Observation:</u> Since t is arc length: $\frac{ds}{dt} \equiv 1$.

$$ds = \underbrace{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}_{1}\; dt$$

So $\boxed{\dot{x}(t)^2 + \dot{y}(t)^2 \equiv 1}$

Now just do it!

$$A = \frac{1}{2}\int_\gamma -y\,dx + x\,dy$$

$$= \frac{\pi}{2\pi}\int_{-\pi}^{\pi} -y(t)\,\dot{x}(t) + x(t)\,\dot{y}(t)\; dt$$

$$= \pi\left(\frac{1}{2\pi}\langle -y, \dot{x}\rangle + \frac{1}{2\pi}\langle x, \dot{y}\rangle\right)$$

$$= \pi \sum_{-\infty}^{\infty} \Big(-\hat{y}(n) \overline{\underbrace{\hat{\dot{x}}(n)}_{in\hat{x}(n)}} + \hat{x}(n) \overline{\underbrace{\hat{\dot{y}}(n)}_{in\hat{y}(n)}} \Big)$$

$$= \pi i \sum_{-\infty}^{\infty} \Big(n\hat{y}(n)\overline{\hat{x}(n)} - n\hat{x}(n)\overline{\hat{y}(n)} \Big)$$

Next: $\displaystyle\int_{-\pi}^{\pi} \underbrace{\dot{x}(t)^2 + \dot{y}(t)^2}_{=1} \, dt = 2\pi$

$$\underset{\text{Parseval's}}{=} \quad 2\pi \sum_{-\infty}^{\infty} \Big(|in\hat{x}(n)|^2 + |in\hat{y}(n)|^2 \Big)$$

Divide out 2π's. Get

$$\pi \cdot 1 = \pi \sum_{-\infty}^{\infty} n^2 \Big(|\hat{x}(n)|^2 + |\hat{y}(n)|^2 \Big)$$

Finally, Want $A \leq \pi$

$0 \leq \pi - A$

$$\pi - A = \pi \sum_{-\infty}^{\infty} n^2 \Big(|\hat{x}(n)|^2 + |\hat{y}(n)|^2 \Big)$$
$$- \pi \sum_{-\infty}^{\infty} in \Big(\hat{y}(n)\overline{\hat{x}(n)} - \hat{x}(n)\overline{\hat{y}(n)} \Big)$$

Simple algebra will show that this a sum of squares!

So ≥ 0! Isop. ineq. ✓

If $\tilde{\pi} - A = 0$. Each of squared terms must $= 0$.

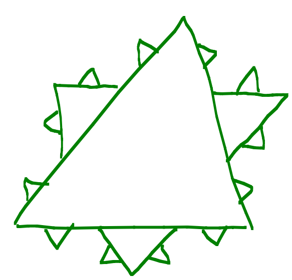
Will see $x(t), y(t)$ must be

simple combos of $\sin 1 \cdot t$, $\cos 1 \cdot t$

making γ a circle!

Great book: Dym & McKean, Applications of Fourier analysis.

How bad can isop. ineq get?



etc.

Area $< \infty$

Peremeter $\longrightarrow \infty$!

Sierpinski snowflake