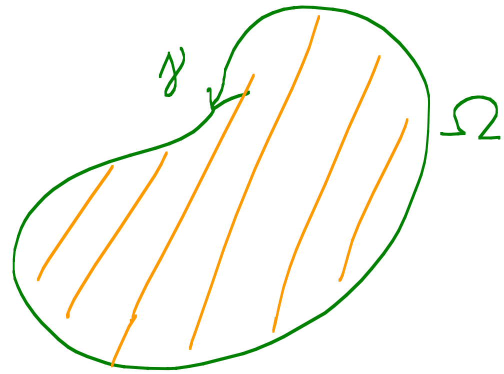
Lesson 22 Isoparametric inequality via Fourier series

HWK 6 on Home page
due Fri after break



$$A \le \frac{L^2}{4\pi}$$

L = length of simple closed C^2-smooth curve γ.

Take $L = 2\pi$.
Show $A \le \pi$
$\pi - A \ge 0$.

$$A = \frac{1}{2}\int_\gamma -y\,dx + x\,dy$$

γ: $(x(t), y(t))$
$-\pi \le t \le \pi$
t = arc length

$$= \frac{1}{2}\int_{-\pi}^{\pi} -y(t)\dot{x}(t) + x(t)\dot{y}(t)\,dt$$

Parseval's: $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

γ $\underline{\underline{C^2}}$-smooth. $x(t) = \sum_{n=-\infty}^{\infty} \hat{x}(n)e^{int}$ ← uniform conv!

$\dot{x}(t) = \sum_{n=-\infty}^{\infty} \underbrace{\hat{\dot{x}}(n)}_{in\hat{x}(n)} e^{int}$ ← unif too!

$$A = \pi \sum_{-\infty}^{\infty}\left(-\hat{y}(n)\overline{[in\hat{x}(n)]} + \hat{x}(n)\overline{[in\hat{y}(n)]}\right)$$

$$\boxed{A = \pi \sum_{-\infty}^{\infty} in\left(\hat{y}(n)\overline{\hat{x}(n)} - \hat{x}(n)\overline{\hat{y}(n)}\right)}$$

complex conjugates

Note: $\overline{(zw)} = \bar{z}\,\bar{w}$ $\qquad \overline{(z\bar{w})} = \bar{z}\,w$

$$z+\bar{z} = (x+iy)+(x-iy) = 2x = 2\,\mathrm{Re}\, z$$

$$z-\bar{z} = 2iy = i\,2\,\mathrm{Im}\, z$$

$$A = \pi \sum_{-\infty}^{\infty} n\,2\left(\mathrm{Im}\,\hat{x}(n)\overline{\hat{y}(n)}\right)$$

Trick: t arc length: $(\dot{x}(t), \dot{y}(t))$ has unit length.

$$\dot{x}(t)^2 + \dot{y}(t)^2 \equiv 1$$

So $$\int_{-\pi}^{\pi} \dot{x}(t)^2 + \dot{y}(t)^2\, dt = 2\pi$$

$$= 2\pi \sum_{-\infty}^{\infty} \left(|in\hat{x}(n)|^2 + |in\hat{y}(n)|^2 \right)$$

Parseval's

$$\boxed{\pi = \sum_{-\infty}^{\infty} n^2 \left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2 \right)}$$

$\pi - A$ = Sum of squares!

$$= \pi \sum_{-\infty}^{\infty} \left(n^2\left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2\right) - in\left(\hat{y}(n)\overline{\hat{x}(n)} - \hat{x}(n)\overline{\hat{y}(n)}\right) \right)$$

Write $\hat{x}(n) = \alpha_n + i\beta_n$

$\hat{y}(n) = a_n + i\, b_n$

Algebra: $$\pi - A = \pi \sum_{-\infty}^{\infty} \Big[n^2 (\alpha_n^2 + \beta_n^2 + a_n^2 + b_n^2) - 2n(\beta_n a_n - \alpha_n b_n) \Big]$$

$$= \pi \sum_{-\infty}^{\infty} \Big[(\underline{n\alpha_n + b_n})^2 + (\underline{n\beta_n - a_n})^2 + (n^2-1)(\underline{a_n^2 + b_n^2}) \Big]$$

Sum of squares! So ≥ 0. Iso. ineq ✓

What if $\Sigma = 0$? For $|n|>1$: $a_n = 0$, $b_n = 0$

Blue ones show $\alpha_n = 0$, $\beta_n = 0$ $|n|>1$.

$n=1$ terms: $1\cdot\alpha_1 + b_1 = 0$ $\boxed{\begin{matrix} b_1 = -\alpha_1 \\ a_1 = \beta_1 \end{matrix}}$

$1\cdot\beta_1 - a_1 = 0$

$n=-1$ terms: $(-1)\alpha_{-1} + b_{-1} = 0$ $b_{-1} = \alpha_{-1}$

$(-1)\beta_{-1} - a_{-1} = 0$ $a_{-1} = -\beta_{-1}$

Because $x(t)$, $y(t)$ real $\hat{x}(-n) = \overline{\hat{x}(n)}$

$$\hat{x}(-n) = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\, e^{-i(-n)t}\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\, e^{int}\, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x(t)\, e^{-int}\, dt$$

$$= \hat{x}(n)$$

Bottom line:

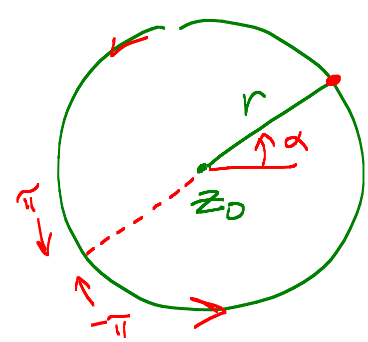
$$\begin{cases} x(t) = \alpha_0 + (\alpha_1 + i\beta_1) e^{it} + (\alpha_{-1} + i\beta_{-1}) e^{-it} \\ y(t) = a_0 + (a_1 + i b_1) e^{it} + (a_{-1} + i b_{-1}) e^{-it} \end{cases}$$

$$\begin{cases} x(t) = \alpha_0 + 2\alpha_1 \cos t - 2\beta_1 \sin t \\ y(t) = a_0 + 2\beta_1 \cos t + 2\alpha_1 \sin t \end{cases}$$

Circle?

Yes! $$x(t) + i y(t) = \underbrace{(\alpha_0 + i a_0)}_{z_0} + \underbrace{2(\alpha_1 + i\beta_1)}_{re^{i\alpha}} \underbrace{(\cos t + i \sin t)}_{e^{it}}$$

$$z(t) = z_0 + r e^{i(t+\alpha)}$$

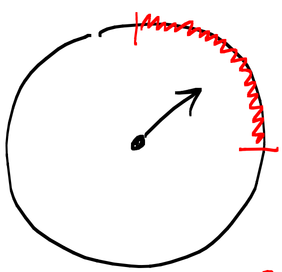


Read Chapter 4.

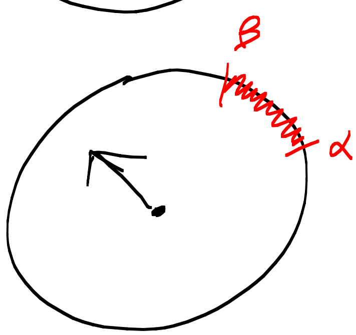
Next biggy: Weyl's Equidistribution Theorem.

Gets used to create "pseudo-random" numbers

Thm: If α is an irrational multiple of π, then $e^{in\alpha}$ are equidistributed on the unit circle.



$$\frac{\#\text{spins in red}}{\text{Total }\#\text{ spins}} \to \frac{1}{4}$$



$$\frac{\#\text{spins in }(\alpha,\beta)}{\text{Total }\#\text{spins}} \to \frac{\beta-\alpha}{2\pi}$$

$$\frac{\#\text{ times } e^{in\alpha} \text{ lands in } (\alpha,\beta)}{N} \to \frac{\beta-\alpha}{2\pi}$$

Herman Weyl (Vile)

Andre Weil (Vay)

Friday: Square drum problem.