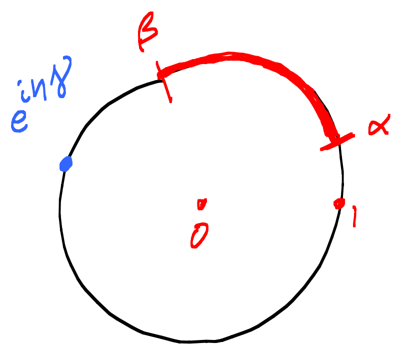
Lesson 23 Weyl's equidistribution theorem



γ an irrational multiple of π

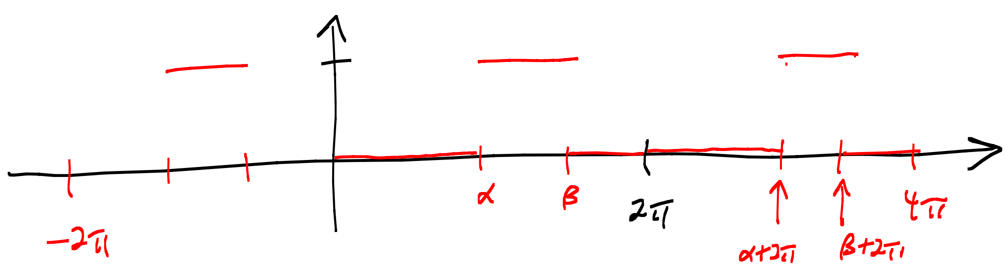
Then $e^{in\gamma}$ are equidistributed on the unit circle $C_1(0)$.

$$R_N = \frac{\#\text{ times } e^{in\gamma} \text{ falls in } (\alpha,\beta) \text{ are } n=1,2,\dots,N}{N} \xrightarrow[N\to\infty]{} \frac{\beta-\alpha}{2\pi}$$

Pf: $\chi(t) = \chi_{(\alpha,\beta)} = \begin{cases} 1 & e^{it} \in \{e^{i\theta} : \alpha < \theta < \beta\} \\ 0 & \text{otherwise} \end{cases}$

2π-periodic on $\mathbb{R}$

$$= \begin{cases} 1 & t \in (\alpha,\beta) \text{ modulo } 2\pi \\ 0 & \text{otherwise} \end{cases}$$



Abuse notation: $\chi(t) = \chi(e^{it})$

$$R_N = \frac{1}{N}\sum_{n=1}^{N} \chi(e^{in\gamma})$$

Aha! Looks like a Riemann sum

$$= \frac{1}{2\pi}\underbrace{\left(\frac{2\pi}{N}\right)}_{\Delta t}\sum_{n=1}^{N}\chi(e^{in\gamma}) \xrightarrow{\text{hope}} \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi\,dt = \frac{\beta-\alpha}{2\pi} \checkmark$$

Weyl's lemma If f is continuous and 2π-periodic,

then
$$\frac{1}{N}\sum_{n=1}^{N} f(e^{in\gamma}) \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

Pf: Step 1: True for f = trig polys

$$= A_0 + \sum_{n=1}^{N}\left(A_n \cos nt + B_n \sin nt\right)$$

(A's, B's are anything, not necessarily the Fourier coeff).

Step 0: True for $f(t) \equiv 1$:

$$R_N = \frac{\overbrace{1+1+1+\cdots+1}^{N\text{-times}}}{N} = \frac{N}{N} = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dt = 1 \checkmark$$

Step 0.5: True for other basis functions in trig polys:

$$f_m(t) = e^{imt}$$

$$f_m(e^{in\gamma}) = e^{im(n\gamma)} = \left(e^{im\gamma}\right)^n$$

$$\frac{f_m(e^{i\gamma}) + f_m(e^{i2\gamma}) + \cdots + f_m(e^{iN\gamma})}{N}$$

$$= \frac{(e^{im\gamma})^1 + (e^{im\gamma})^2 + \cdots + (e^{im\gamma})^N}{N}$$

$$= \frac{e^{im\gamma}}{N}\left[\underbrace{1 + (e^{im\gamma}) + \cdots + (e^{im\gamma})^{N-1}}_{\frac{1-(e^{im\gamma})^{(N-1)+1}}{\underbrace{1-e^{im\gamma}}}}\right]$$

Aha! $e^{i\theta}$ is only $=1$ where $\theta = 2\pi n$, $n \in \mathbb{Z}$

$m\gamma \neq 2\pi n$ for all n!

$$|R_N| \leq \frac{1}{N}\,\frac{1 + |\overbrace{e^{im\gamma N}}^{=1}|}{|1-e^{im\gamma}|} \leq \frac{2}{N\cdot|1-e^{im\gamma}|} \to 0 \text{ as } N\to\infty$$

and: $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imt}\,dt = 0$ for $m = \pm1, \pm2, \pm3, \ldots$

Step 0.75: True for 1, e^{imt} $m = \pm1, \pm2, \ldots, \pm N$

$\Longrightarrow$ true for linear combos.

So true for trig polys.

Step 2 Use Féjer's to see true 2π-periodic continuous $f(t)$: Let $\varepsilon > 0$.

Féjer's says there is a trig poly $P(t)$ such

$$|f(t) - P(t)| < \frac{\varepsilon}{3} \quad \text{for all } t.$$

$$\mathcal{E}_N = \left| \frac{1}{N} \sum_{n=1}^{N} f(e^{in\gamma}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt \right|$$

$$= \left| \underbrace{\frac{1}{N} \sum_{n=1}^{N} \left(f(e^{in\gamma}) - P(e^{in\gamma}) \right)}_{T_1} + \left(\underbrace{\frac{1}{N} \sum_{n=1}^{N} P(e^{in\gamma}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)\, dt}_{T_2} \right) + \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(P(t) - f(t) \right) dt}_{T_3} \right|$$

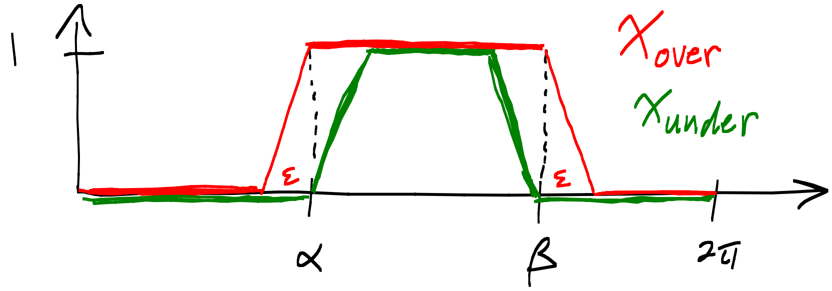
$$|T_1| < \frac{1}{N} \sum_{1}^{N} \frac{\varepsilon}{3} = \frac{\varepsilon}{3}$$

Can make $|T_2| < \frac{\varepsilon}{3}$ if $N > N_0$ because Lemma true for P.

$$|T_3| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|P(t) - f(t)|}_{< \frac{\varepsilon}{3}}\, dt < \frac{1}{2\pi} \cdot \frac{\varepsilon}{3} \cdot 2\pi = \frac{\varepsilon}{3}.$$

So $|\mathcal{E}_N| \le |T_1| + |T_2| + |T_3| < 3 \cdot \frac{\varepsilon}{3}$ if $N > N_0$

<u>Last step in Weyl's proof:</u>



$$\chi_u \le \chi \le \chi_o$$

$$\frac{1}{N}\sum_{n=1}^{N}\chi_u(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N}\chi(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N}\chi_o(e^{in\gamma})$$

Lemma $\downarrow$ $\qquad\qquad$ $\downarrow$ as $N\to\infty$.

$$\frac{\beta-\alpha-\varepsilon}{2\pi} \qquad\qquad \frac{\beta-\alpha+\varepsilon}{2\pi}$$

Finally, let $\varepsilon \to 0$ to see that $\frac{1}{N}\sum_{1}^{N}\chi(e^{in\gamma}) \to \frac{\beta-\alpha}{2\pi}$

as $N\to\infty$. ✓

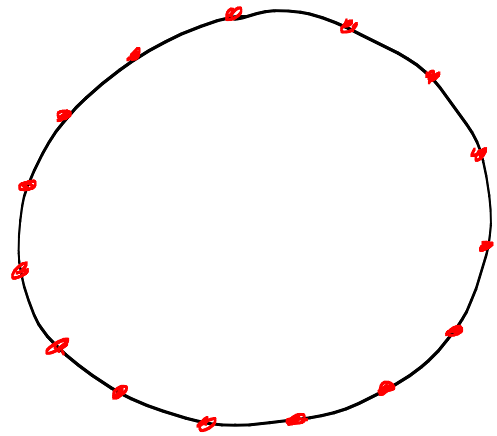
<u>Hmmm</u>: Lemma shows formula true for <u>step fcns</u>.

Riemann integrable fcns are fcns that can be "well approximated" by step fcns.

Conclude formula true for R. integrable fcns.

What if $\gamma = \frac{p}{q}\pi$? ($\frac{p}{q}$ rational)

$e^{in\left(\frac{p}{q}\pi\right)}$:



Repeats!

Computers throwing darts in $[0,1]$.

Get two gigantic primes p, q.

Take big N from clock on computer.

"Throw a dart:" $\underbrace{N\cdot\frac{p}{q} \bmod \mathbb{Z}}_{\text{decimal part.}}$