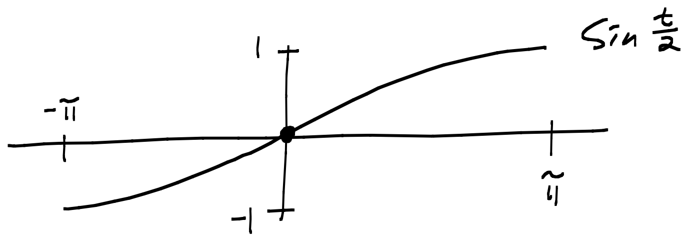
Lesson 24 Exam 1 sol'ns, vibrating rectangular drums

Prob 4: $\displaystyle\int_{-\pi}^{\pi} \frac{\cos Nt}{\sin\frac{t}{2}}\left((x-t)^2 - x^2\right)dt \longrightarrow 0$ as $N\to\infty$



$$g(t) = \frac{(x-t)^2 - x^2}{\sin\frac{t}{2}} \sim \frac{0}{0}$$

at $t=0$

$$\lim_{t\to 0} g(t) = \lim_{t=0} \frac{2(x-t)(-1)}{\frac{1}{2}\cos\frac{t}{2}} = -4x$$

Aha! Make $g(0) = -4x$, so it's continuous on $[-\pi, \pi]$.

$$\int = \pi\cdot\left(\text{Fourier cosine coeff } a_N\right)$$

Either Bessel's ineq (or Riemann-Lebesgue lemma) for continuous g shows that $a_N \to 0$. ✓

Nice problem because

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} D_N(t)\, f(x-t)\,dt - \underbrace{\left(\int_{-\pi}^{\pi} D_N(t)\,dt\right)}_{1} f(x)$$

$$= \int_{-\pi}^{\pi} \underbrace{\frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}}_{D_N(t)} \Big[\underbrace{f(x-t) - f(x)}_{\text{like } (x-t)^2 - x^2}\Big]$$

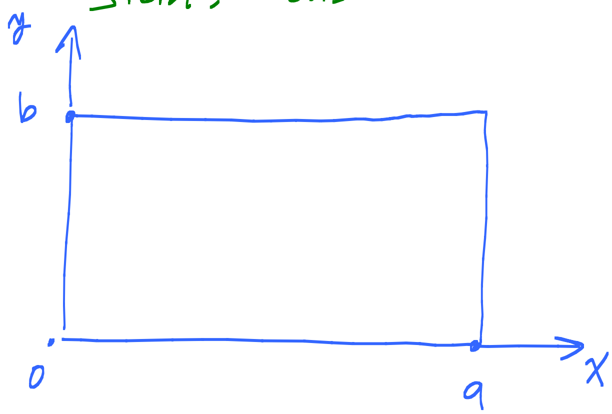
$f(t) = t^2$

$$\sin(N+\tfrac{1}{2})t = \cos Nt \,\sin\tfrac{t}{2} + \sin Nt \,\cos\tfrac{t}{2}$$

1. $$\sin 3\theta = \frac{1}{2i}\left(e^{i3\theta} - e^{-i3\theta}\right)$$
$$= \frac{1}{2i}\left[\left(e^{i\theta}\right)^3 - \left(e^{-i\theta}\right)^3\right]$$
$$= \frac{1}{2i}\left[\left(\cos\theta + i\sin\theta\right)^3 - \left(\cos\theta - i\sin\theta\right)^3\right]$$

2. <u>Not</u> a theorem that (Fourier series for f) converges to $f(x)$ uniformly ... or even pointwise!

Stein: Constructs an example where it fails.



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Take $c = 1$.

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

$$XYT'' = X''YT + XY''T \leftarrow$$ divide by XYT

$$\boxed{\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}} = \lambda, \text{ a const.}$$

(Let $t = \frac{1}{2}$).

$$\boxed{\frac{X''}{X} = \lambda - \frac{Y''}{Y}} = \mu, \quad \text{a const too!}$$

B.C.'s Drum is pinned down on edges.

Translates to $X(0)=0,\ X(a)=0$

$Y(0)=0,\ Y(b)=0$

<u>x - Problem</u>: $X'' - \mu X = 0$

$X(0)=0,\ X(a)=0$

Know
$$\boxed{\begin{aligned} \mu_n &= -\left(\frac{n\pi}{a}\right)^2 \\ X_n(x) &= \sin\frac{n\pi x}{a} \end{aligned}}$$

<u>Y - problem</u>: $\lambda - \dfrac{Y''}{Y} = \mu_n$

$$\boxed{\begin{aligned} Y'' - (\lambda - \mu_n) Y &= 0 \\ Y(0)=0,\ Y(b)&=0 \end{aligned}}$$

Know $\lambda - \mu_n = -\left(\dfrac{m\pi}{b}\right)^2, \quad Y_m = \sin\dfrac{m\pi}{b}y$

$$\lambda_{nm} = \mu_n - \left(\frac{m\pi}{b}\right)^2$$

$$\lambda_{nm} = -\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2$$

Back to T-problem: $T'' - \lambda T = 0$.

$$T_{nm}(t) = A_{nm} \cos \omega_{nm} t + B_{nm} \sin \omega_{nm} t$$

where $\omega_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$

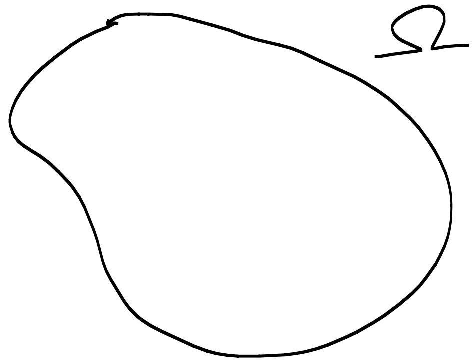
Basic sol^{n}s:

$$u_{nm}(x,y,t) = \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\left(A_{nm} \cos \omega_{nm} t + B_{nm} \sin \omega_{nm} t\right)$$

$$u(x,y,t) = \sum_{n,m} \text{basic sol}^n\text{s}.$$

Square drum: $\pi \times \pi$. Frequencies $\sqrt{n^2 + m^2}$ Ugh!

Fun question: Rectangular drum $a \times b$ such that Area $= ab = A$? Which drum has the lowest pitch: Lowest freq: $\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$ ← lowest possible for square case!



Eigenfunctions for Δ:

$$\begin{cases}\Delta u = \lambda u \\ u|_{b\Omega} = 0\end{cases}$$

Drum $\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$

λ's are "spectrum" of Ω.

Can't hear the shape of a drum!