Review for Exam 1, in class Wed March 4

1. Suppose that $f(x)$ is a C^1-smooth function on $[a, b]$. Use integration by parts to show that

$$I = \int_a^b \underbrace{f(x)}_{u} \underbrace{\cos(Nx)\, dx}_{dv}$$

$du = f'(x)\,dx$

$v = \frac{1}{N} \sin Nx$

tends to zero as the positive integer N tends to infinity.

$$I = uv\Big|_a^b - \int_a^b v \, du = \frac{1}{N}\Big(f(x) \sin Nx\Big)_a^b - \frac{1}{N}\int_a^b f'(x) \sin Nx \, dx$$

$$|I| \le \frac{1}{N}\left[2 \max_{[a,b]} |f| + \left(\max_{[a,b]} |f'|\right)(b-a)\right] \to 0 \text{ as } N \to \infty.$$

$$\left| \int_a^b h(x)\,dx \right| \le \int_a^b \underbrace{|h(x)|}_{\le \max_{[a,b]} |h|} dx \le \left(\max_{[a,b]} |h|\right)(b-a)$$

2. Estimate the integral in problem one in terms of the maximum value M of $|f(x)|$ on $[a, b]$.

$$\left| \int_a^b f(x) \cos Nx \, dx \right|$$

$$\leq \left(\max_{[a,b]} \underbrace{\left| f(x) \cos Nx \right|}_{\leq M} \right) (b-a) \leq M(b-a)$$

3. Show that the integral in problem one goes to zero as $N \to \infty$ when f is merely assumed to be continuous by approximating f by a polynomial and then using your results from problem one and two.

Approx f by a poly $p(x)$. Let $\varepsilon > 0$.

Assume $|f(x) - p(x)| < \frac{\varepsilon}{2(b-a)}$ on $[a,b]$. (Weierstraß)

$$\int_a^b f(x) \cos Nx \, dx$$

$$= \underbrace{\int_a^b [f(x) - p(x)] \cos Nx \, dx}_{} + \underbrace{\int_a^b p(x) \cos Nx \, dx}_{}$$

Prob 2 estimate:

$$\left| \cdots \right| \le \left(\max_{[a,b]} |f - p| \right)(b-a)$$

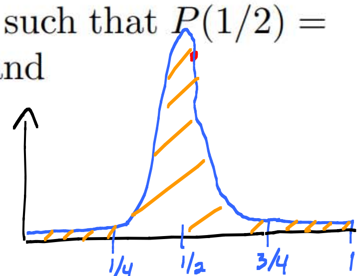
$$< \frac{\varepsilon}{2}$$

$\to 0$ as $N \to \infty$ by Prob. 1.

$$< \frac{\varepsilon}{2}$$

✓

4. Suppose $P(x)$ is a positive continuous function on $[0,1]$ such that $P(1/2) = 10^4$, and $P(x) < 1/100$ when x is in $[0,1/4]\cup[3/4,1]$, and



$$\int_0^1 P(x)\,dx = 1.$$

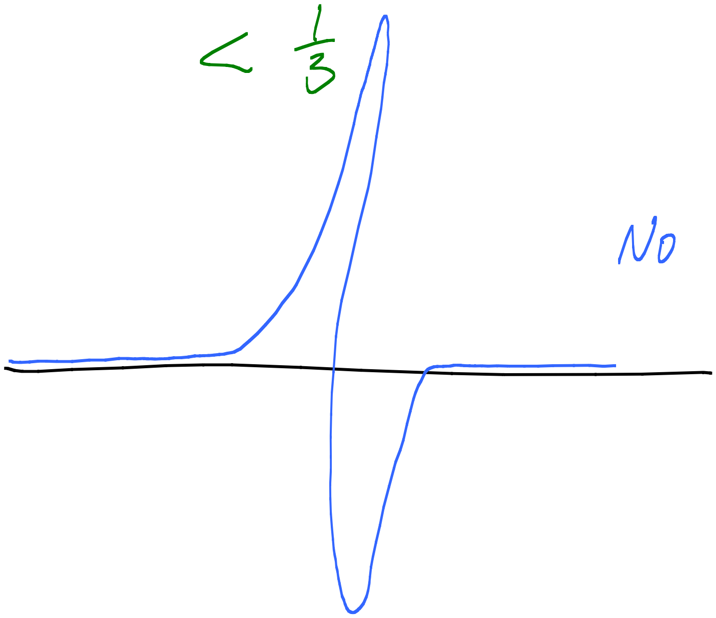
If f is a continuous function such that $|f(x)| < 17$ on $[0,1]$, but $|f(x)| < 1/3$ on $[1/4,3/4]$, estimate how large

$P(x) \geq 0$

$$\left|\int_0^1 P(x)f(x)\,dx\right| \le \underbrace{\left(\int_{[0,\frac14]} + \int_{[\frac34,1]}\right) \underbrace{P(x)}_{<\frac{1}{100}}\underbrace{|f(x)|}_{<17}\,dx}_{<\frac{17}{100}\underbrace{(\text{lengths})}_{\frac14+\frac14}\ <\ \frac{17}{200}}$$

could be. What is your best estimate for the bound if the assumption that P is positive is dropped?

$$+\underbrace{\int_{[\frac14,\frac34]} P(x)\underbrace{|f(x)|}_{<\frac13}\,dx \le \frac13\underbrace{\int_{1/4}^{3/4} P(x)\,dx}_{\le \int_0^1 P(x)\,dx = 1}}_{<\frac13}$$



No estimate if $P \geq 0$ is dropped

1. Explain why

$$\int_{-\pi}^{\pi} \underbrace{\underbrace{e^{-x^2}}_{\text{even}} \underbrace{\sin 3x}_{\text{odd}}}_{\text{odd}}\, dx = 0$$

without trying to find an antiderivative (which is impossible).

Important: Interval is symmetric about origin.

Facts: Even fcns have no sine terms in their F. Series.

Odd have no cosine terms.

1. Show that the three functions x^7, $\cos 3x$, and 1 are pairwise orthogonal on $[-\pi, \pi]$.
If

$$x^7 f(x) = \left(c_0 + c_1 x^7 + c_2 \cos 3x,\right) x^7$$

write an integral formula for c_1 involving $f(x)$ and the given functions.

$$\int_{-\pi}^{\pi} \underbrace{x^7}_{\text{odd}} \underbrace{\cos 3x}_{\text{even}} \, dx = 0$$

$$\int_{-\pi}^{\pi} \underbrace{x^7 \cdot 1}_{\text{odd}} \, dx = 0$$

$$\int_{-\pi}^{\pi} 1 \cdot \cos 3x \, dx = 0 \quad \text{easy.}$$

$$\int_{-\pi}^{\pi} x^7 f(x)\,dx$$

$$= c_0 \cancel{\int_{-\pi}^{\pi} 1\, x^7\, dx}\ 0$$

$$+ c_1 \int_{-\pi}^{\pi} x^{14}\, dx$$

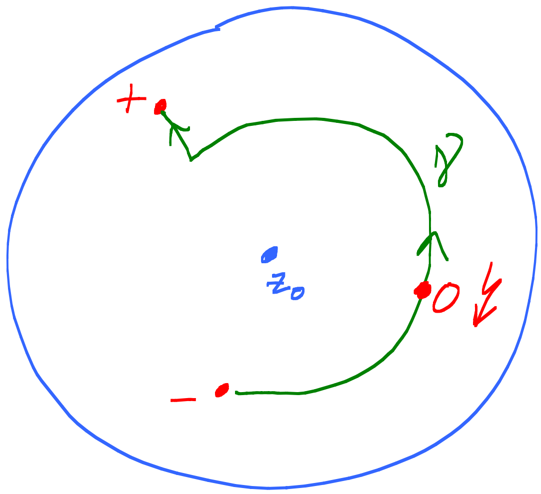
$$+ c_2 \cancel{\int_{-\pi}^{\pi} x^7 \cos 3x\, dx}\ 0$$

$$\int_{-\pi}^{\pi} x^{14} dx = 2\int_{0}^{\pi} x^{14} dx = 2\,\frac{\pi^{15}}{15}$$

$$\int_{-\pi}^{\pi} x^7 f(x)\, dx = c_1 \cdot \frac{2\pi^{15}}{15}$$

Get c_1

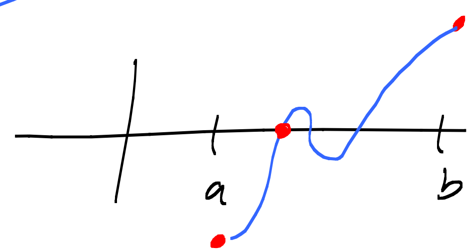
4. Prove that a continous real valued function on an open disc that has a single zero at the center of the disc must be either always positive away from the zero, or always negative. Use this fact to show that a continuous function that satisfies the averaging property cannot have an isolated zero. Why does this imply that harmonic functions cannot have isolated zeroes?



$u(z_0) = 0$

z_0 is the <u>only</u> zero of u

$\gamma: z(t),\ a \le t \le b \qquad u(z(t))$



Intermediate value thm.

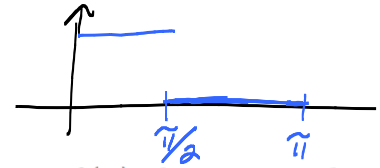
z_0 isolated zero of (real valued) harmonic fcn

$$0 = u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + re^{it})}_{\substack{\text{always } >0 \\ \text{or} \\ \text{always } <0}}\, dt \quad \begin{matrix} >0 \\ \text{or} \\ <0 \end{matrix}$$

Contradiction! So u cannot have an <u>isolated</u> zero.

4. Assume that

$$S(x) = \sum_{n=1}^{\infty} B_n \sin nx$$



is the sum of the Fourier sine series on $[0, \pi]$ for the function $f(x)$ that is equal to one on the interval $[0, \pi/2)$ and zero on $[\pi/2, \pi]$.

a) Write an integral formula for B_n and evaluate the integral (but don't simplify it or try to see a pattern).

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} 1\cdot \sin nx\,dx + 0$$

b) Graph the function that the Fourier sine series converges to on $[-3\pi, 3\pi]$, being careful at jumps.

c) Evaluate the sum

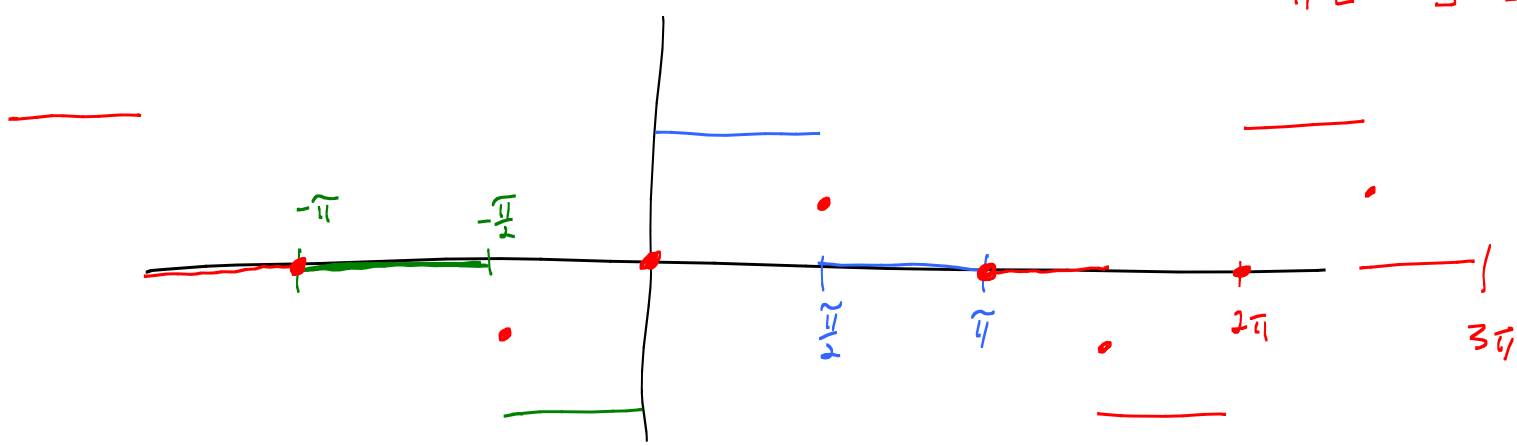
$0 \quad \cancel{2a_0^2 + \sum_{n=1}^{\infty} a_n^2} +$ $\displaystyle\sum_{n=1}^{\infty} {B_n}^2.$

Parseval's

$$\sum_{n=1}^{\infty} {B_n}^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\tilde{f}(x)|^2\,dx = \frac{2}{\pi}\int_0^{\pi} |f(x)|^2\,dx = \frac{2}{\pi}\left[1\cdot\frac{\pi}{2}\right] = 1$$

extension of f

Hint: The Fourier sine series is equal to the full Fourier series of the function you graphed in part (b).



Take odd 2π-periodic extension of f.

Fourier sine series for f on $[0, \pi]$ is the full Fourier series for the odd 2π-periodic extension. Read off series converges to smooth parts and mid points of jumps.

$$1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}$$

$z = -w$

$$1 - w + w^2 + \cdots + (-1)^N w^N = \frac{1 - (-1)^{N+1} w^{N+1}}{1 + w}$$

$$w = e^{i\theta} = \cos\theta + i \sin\theta$$

$$w^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$1 - \sin\theta + \sin 2\theta - \sin 3\theta + \cdots + (-1)^N \sin N\theta$$

$$= \operatorname{Im}\left[\frac{1 - (-1)^{N+1}\left(e^{i(N+1)\theta}\right)}{1 + e^{i\theta}}\right]$$

Sturm-Liouville: $X'' \pm \lambda X = 0$ ← know how to solve.

B.C.'s

Cases $\lambda < 0$, $\lambda = 0$
$\lambda > 0$.