MA 428 Fourier Analysis

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MATH 750
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HWK	100	pts
2 Midterm	200	pts
Final exam	200	pts
	500	pts

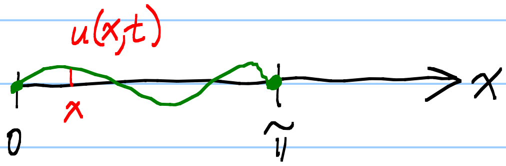
Power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Fourier series: $f(x) = A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$

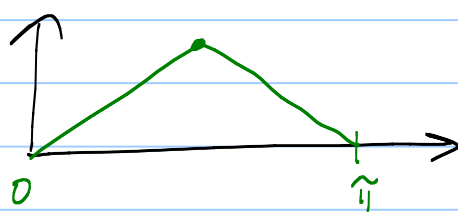
Bernoulli vs Bernoulli: Vibrating string problem: Harpsichord



Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Time $t=0$;



$f(x)$ = "pluck shape"

Initial conditions: (IC)

$$\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

← pluck, not hit,

Boundary conditions: (BC)

$$\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

Jacob's solution: $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right]$

Johann's solution: Try $u(x,t) = X(x)T(t)$

(separation of variables)

Assume $c=1$.

Plug into PDE: $\frac{\partial^2}{\partial t^2}\left[X(x)T(t)\right] = \frac{\partial^2}{\partial x^2}\left[X(x)T(t)\right]$

$$X(x)T''(t) \overset{\text{want}}{=} X''(x)T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

[Let $x=\frac{\pi}{2}$. See RHS = const. Now so is LHS.]

Aha! Get ODEs $\begin{cases} X'' - \lambda X = 0 \\ T'' - \lambda T = 0 \end{cases}$

Boundary cond: $u(0,t) = X(0)T(t) \equiv 0$

$u(\pi,t) = X(\pi)T(t) \equiv 0$

Hmmm. $T(t) \equiv 0$ implies BC, but yields only $\equiv 0$ sol^n.

So need: $\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$

X problem: $X'' - \lambda X = 0$ (Sturm-Liouville prob)

BC $\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$

Initial conditions: later

MA 262: $X'' - \lambda X = 0$ — 2^{nd} order linear homog ODE with const^{t} coeff.

Idea: Try $X(x) = e^{rx}$

$$r^2 e^{rx} - \lambda e^{rx} \overset{\text{want}}{=} 0$$

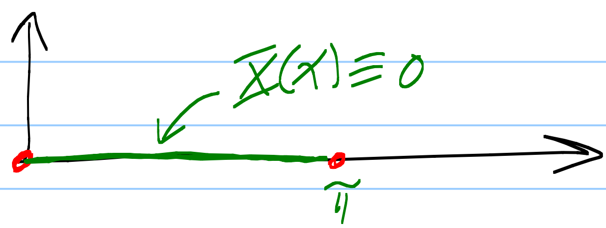
$$\underbrace{(r^2 - \lambda)}_{=0} \underbrace{e^{rx}}_{\text{never } 0} = 0$$

Need $r^2 = \lambda$.

Case $\lambda = 0$: $X'' = 0$

$$X' = c_1$$

$$\boxed{X = c_1 x + c_2} \quad \text{line}$$



Ouch! Only get $u \equiv 0$ sol^{n}.

Case $\lambda > 0$: Write $\lambda = k^2$ $(k > 0)$.

$$r^2 = \lambda = k^2 \qquad \text{Two sol}^{n}\text{'s}: e^{kx}, e^{-kx}$$

$$r = \pm k$$

$$X(x) = c_1 e^{kx} + c_2 e^{-kx} \quad (\text{linear, homog})$$

E. & U. Thm: Given initial conds $\begin{cases} X(0) = A \\ X'(0) = B \end{cases}$, there is a

unique sol^n to ODE satisfying them. Wronskian $\neq 0$
$\Rightarrow$ we have $\text{gen}^l \text{ sol}^n$.

BC $\quad X(0) = c_1 e^{k \cdot 0} + c_2 e^{-k \cdot 0} = 0$

want

$$c_1 + c_2 = 0 \qquad \boxed{c_2 = -c_1}$$

$$X(\pi) = c_1 e^{k\pi} - c_1 e^{-k\pi} = 0$$

$$c_1 \underbrace{(\overset{}{\underset{>1}{e^{k\pi}}} - \underset{<1}{e^{-k\pi}})}_{\neq 0} = 0$$

Ouch! $c_1 = 0$, $c_2 = -c_1 = 0$. Zero sol^n again.

Case $\lambda < 0$: $\lambda = -k^2$

$$r^2 - (-k^2) = 0$$
$$r^2 + k^2 = 0$$
$$r = \pm k i$$

Euler: e^{ikx} = power series = $\cos kx + i \sin kx$

Hmmm: $\frac{d}{dx}\left(e^{ikx}\right) = -k \sin kx + ik \cos kx$
$$= ik\, e^{ikx}$$

$$\frac{d^2}{dx^2}\left(e^{ikx}\right) = (ik)^2 e^{ikx} = -k^2 e^{ikx}$$

Aha! $\underset{u}{\cos kx} + i \underset{v}{\sin kx}$ is a complex sol^n!

$$[u+iv]'' - \lambda[u+iv] = 0$$

$$\underbrace{[u''+\lambda u]}_{=0} + i\underbrace{[v''+\lambda v]}_{=0} = 0$$

Get two real sol^n's,

$$X(x) = c_1 \cos kx + c_2 \sin kx$$