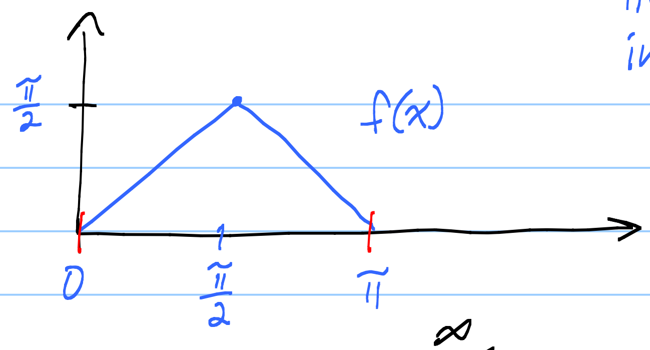
Lesson 3 Fourier sine series

(No class Monday MLK Day) 1/23

HWK 1 due the next Monday, 11:00 pm in Gradescope (linked to in BS)



$$f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ (\pi - x) & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Sol^n $$u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos nt$$

where $$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} x \sin nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx$$

$$A_n\text{'s} = \frac{4}{\pi} \cdot \left(\frac{1}{1^2}, 0, -\frac{1}{3^2}, 0, \frac{1}{5^2}, 0, -\frac{1}{7^2}, 0, \dots \right)$$

Conical bore horns: Euphoniums, Tubas, saxophone, french horn

All A_n's $\neq 0$

Cylindrical bores: trumpets, trombones, clarinets

All $A_{even} = 0$.

Sol^n: $$u(x,t) = \frac{4}{\pi} \left(\frac{1}{1^2} \sin x \cos t - \frac{1}{3^2} \sin 3x \cos 3t + \frac{1}{5^2} \sin 5x \cos 5t - \dots \right)$$

$$u(x,0) = \frac{4}{\pi} \left(\frac{1}{1^2} \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \dots \right)$$

$\stackrel{?}{=} f(x)$

Review: $\varphi_n(x) \to F(x)$ converge uniformly on $[a,b]$

if, given $\varepsilon > 0$, there is an N such that

$|\varphi_n(x) - F(x)| < \varepsilon$ on $[a,b]$ where $n > N$.

Fact: Uniform limits of continuous fcns are continuous.

Def^n: $\varphi_n(x)$ is uniformly Cauchy if, given $\varepsilon > 0$,

there is an N such that $|\varphi_n(x) - \varphi_m(x)| < \varepsilon$

when $n, m > N$ on $[a,b]$.

Fact: Uniformly Cauchy sequences converge uniformly.

Weierstraß M-test: Suppose $\varphi_n(x)$ are continuous on $[a,b]$

and $|\varphi_n(x)| \le M_n$ on $[a,b]$. If $\sum_{n=1}^{\infty} M_n < \infty$,

then $\sum_{n=1}^{\infty} \varphi_n(x)$ converges uniformly to $F(x)$,

which is continuous too.

Pf: $N > M$: $\left| \sum^{N} - \sum^{M} \right| = \left| \sum_{M+1}^{N} \right| \le \underbrace{\sum_{n=M+1}^{N} M_n}_{\text{tail end of conv series}}$

See $\sum^{N} \varphi_n$ is unif Cauchy. ✓

EX: $\left| -\frac{1}{7^2} \sin 7x \right| \le \frac{1}{7^2} \underbrace{\left| \sin 7x \right|}_{\le 1} \le \frac{1}{7^2}$

$$\sum_{n=odd} \frac{1}{n^2} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{= \pi/6} < \int_1^{\infty} \frac{1}{x^2}\, dx < \infty$$

integral test

Consequently, our Fourier sine series converges <u>uniformly</u> to a continuous fcn. MA 428: limit is $f(x)$