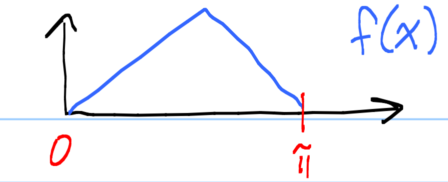
Lesson 4 Fourier sine series, cont., Full Fourier series



$$f(x) \overset{!}{=} \sum_{n=0}^{\infty} A_n \sin nx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

Harpsichord sol^n: $u(x,t) = \sum_{n=0}^{\infty} A_n \underline{\sin nx \cos nt}$

Trig identity: $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

Johann's sol^n => Jacob's

$$u(x,t) = \sum_{n=0}^{\infty} \frac{1}{2} A_n\left[\sin(nx+nt) + \sin(nx-nt)\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} A_n \sin n\underbrace{(x+t)}_{\theta_1} + \sum_{n=0}^{\infty} A_n \sin n\underbrace{(x-t)}_{\theta_2}\right]$$

$$= \frac{1}{2}\left[f(\theta_1) + f(\theta_2)\right]$$

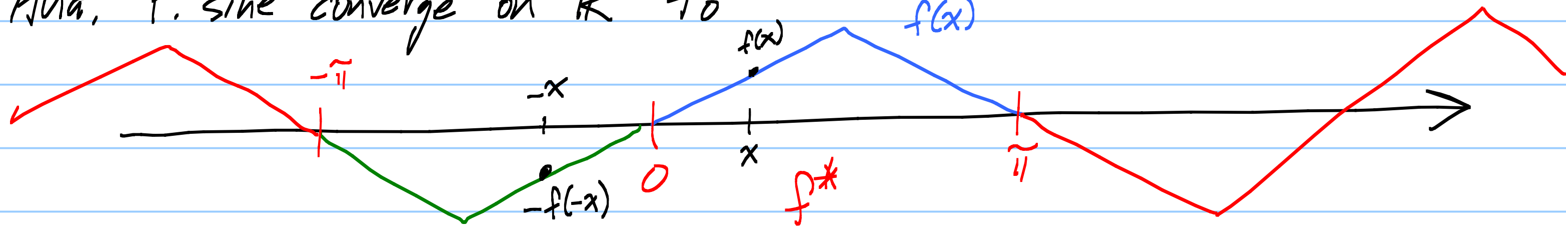
$$= \frac{1}{2}\left[f(x+t) + f(x-t)\right] \qquad \text{Jacob's sol}^n!$$

Hmmm. What happens when $x+t$ and $x-t$ exit $[0,\pi]$?

Important observation about F. sine series:

$$f(x) = \sum_{n=0}^{\infty} A_n \underbrace{\sin nx}_{\text{odd, and } 2\pi\text{-periodic}}$$

Aha! F. sine converge on $\mathbb{R}$ to



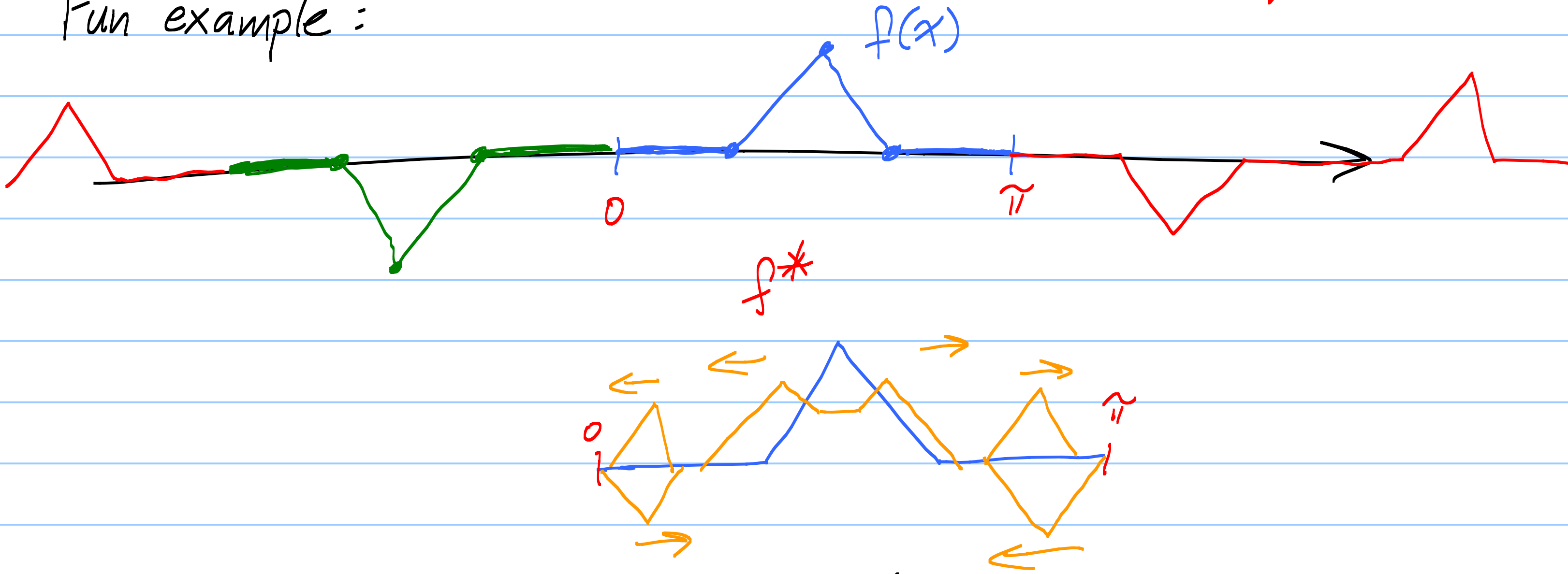
Steps to get f^*: 1) extend f from $[0,\pi]$ to $[-\pi,0]$ as an odd fcn.

2) now extend as a 2π-periodic fcn to $\mathbb{R}$

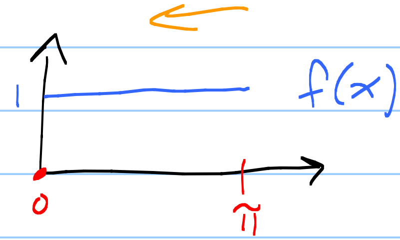
Correct version of Jacob's sol^n:

$$u(x,t)=\frac{1}{2}\Big[\underbrace{f^*(x+t)}_{\text{moves left}}+\underbrace{f^*(x-t)}_{\text{moves right}}\Big]$$

Fun example:



EX: Find F. sine series for



$$A_n=\frac{2}{\pi}\int_0^{\pi}1\cdot\sin nx\,dx=\frac{2}{\pi}\Big[-\frac{1}{n}\cos nx\Big]_0^{\pi}$$

$$=\frac{2}{\pi n}\Big[-\underbrace{\cos n\pi}_{(-1)^n}+\underbrace{\cos 0}_{1}\Big]=\frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$=\frac{2}{\pi}\left(\frac{2}{1},0,\frac{2}{3},0,\frac{2}{5},0,\ldots\right)$$

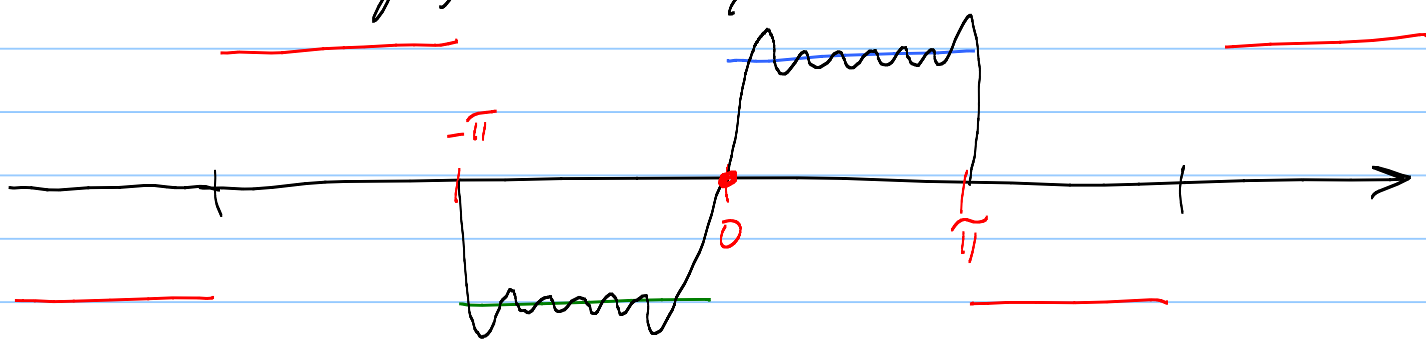
$$= \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f(x) \stackrel{?}{=} \frac{4}{\pi}\left(\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Ouch! $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty$

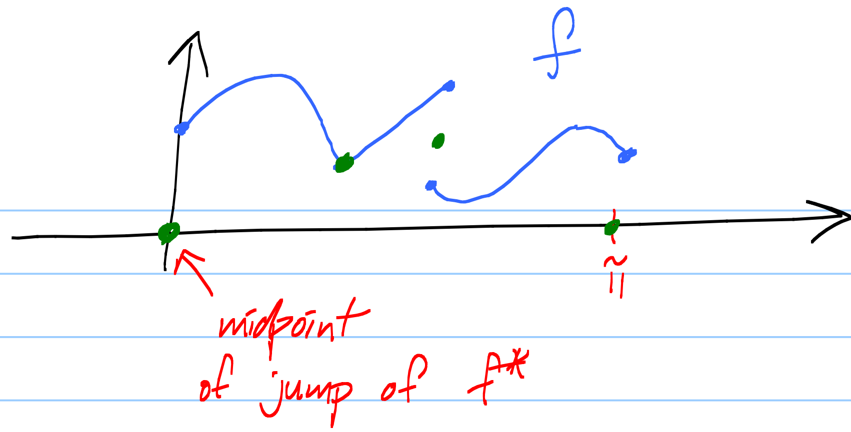
Can't use Weierstraß M-test to conclude that F. series converges (to something).

It <u>does</u> converge, but slowly.



<u>Big fact</u> Fourier sine series converges to f^* (odd 2π-periodic extension) on $\mathbb{R}$. It converges to f^* on intervals where f^* is C^1-smooth. It converges to the <u>midpoint</u> of a jump at jump points. [It converges to zero at each $n\pi$.] It also converges to f^* at "kinks", but more slowly.

EX



Full Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

Key: $\{1, \cos nx, \sin nx\}$ $\quad n = 1, 2, 3, \ldots$

are orthogonal on $[-\pi, \pi]$.

Check:

$$\int_{-\pi}^{\pi} 1 \cdot \underbrace{\cos nx}_{\text{or } \sin nx} \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad \text{for all } n, m$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} 1^2 \, dx = 2\pi$$

What do coeff have to be?

$$\int_{-\pi}^{\pi} f(x)\cos 3x\,dx = \int_{-\pi}^{\pi}\left(a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right)\cos 3x\,dx$$

$$= \underbrace{a_0\int_{-\pi}^{\pi} 1\cdot\cos 3x\,dx}_{=0} + \sum_{n=1}^{\infty}\left(a_n\underbrace{\int}_{\text{all}=0} + b_n\underbrace{\int}_{\text{all}=0}\right)$$

except 1 :

$$= 0 + a_3\underbrace{\int_{-\pi}^{\pi}\cos^2 3x\,dx}_{=\pi} + 0$$

So $a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 3x\,dx$. etc.

Famous formulas:

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx & \leftarrow \text{ave!} \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx & n=1,2,3,\ldots \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx & n=1,2,3,\ldots \end{cases}$$