Lesson 5 Even & odd functions, complex version of Fourier series.

$$f(x) \text{ "=" } a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$\begin{cases} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \longleftarrow \text{ave on } [-\pi, \pi] \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n = 1, 2, \ldots \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad n = 1, 2, \ldots \end{cases}$$

Complex version: $z = x + iy$

$$e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x} \left(\cos y + i \sin y \right)$$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos\theta - i \sin\theta$$

$+$: $\quad e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$-$: $\quad e^{i\theta} - e^{-i\theta} = 2i \sin\theta$

$$\begin{cases} \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \underbrace{\cos nx}_{\frac{e^{inx} + e^{-inx}}{2}} + b_n \underbrace{\sin nx}_{\frac{e^{inx} - e^{-inx}}{2i}} \Big)$$

$$\underbrace{\qquad}_{\frac{1}{2} a_n \left(e^{inx} + e^{-inx} \right)} \qquad \underbrace{\qquad}_{-\frac{1}{2} i\, b_n \left(e^{inx} - e^{-inx} \right)}$$

$\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = -i$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[\underbrace{\left(\frac{a_n}{2} - i\frac{b_n}{2}\right)}_{c_n} e^{inx} + \underbrace{\left(\frac{a_n}{2} + i\frac{b_n}{2}\right)}_{c_{-n}} e^{-inx}\right]$$

$$c_n = \frac{1}{2}(a_n - i b_n) = \frac{1}{\pi}\cdot\frac{1}{2}\int_{-\pi}^{\pi} f(x)\left[\underbrace{\cos nx - i \sin nx}_{e^{-inx}}\right]dx$$

Wow!

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx}$$

c_{-n} = Same formula with $(-n)$ in place of n.

$$c_0 = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \underbrace{e^{-i\cdot 0\cdot x}}_{1}\, dx \quad \checkmark$$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx}$$

Real inner product: $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

but if g is complex valued, then

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx$$

Big fact: $\{e^{inx}\}_{n=-\infty}^{\infty}$ are **orthogonal**!

$$\langle e^{inx}, e^{imx} \rangle = \int_{-\pi}^{\pi} e^{inx} \underbrace{\overline{e^{imx}}}_{e^{-imx}} \, dx$$

$$= \int_{-\pi}^{\pi} \underbrace{e^{i(n-m)x}}_{\cos(n-m)x + i\sin(n-m)x} \, dx$$

$$= \int_{-\pi}^{\pi} \cos(n-m)x \, dx + i \int_{-\pi}^{\pi} \sin(n-m)x \, dx$$

$$= \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$$

Back to real version:

$$f(x) = \underset{\text{even}}{a_0} + \sum_{n=1}^{\infty} \Big(a_n \underbrace{\cos nx}_{\text{even}} + b_n \underbrace{\sin nx}_{\text{odd}} \Big)$$

Fun fact Any fcn = (even fcn) + (odd fcn)

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even!}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd!}}$$

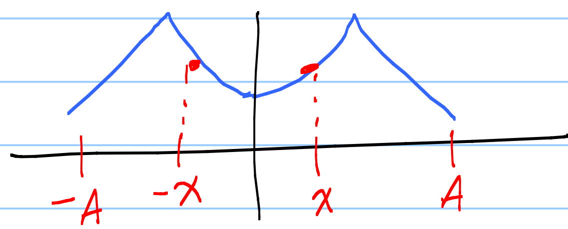
Unique decomposition!

Useful facts If f is odd, all $\overbrace{a_n \text{ terms}}^{\text{all cosine terms } = 0} = 0$.

If f is even, all $\overbrace{b_n \text{ terms}}^{\text{all sine terms } = 0} = 0$.

Why: $$\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos nx}_{\text{even}}}_{\text{odd}} \, dx = 0$$

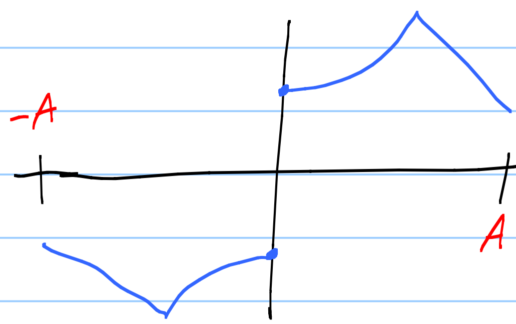
Even:



$f(x)$ even:

$$\int_{-A}^{A} f(x)\,dx = 2\int_0^A f(x)\,dx$$

Odd:



$f(x)$ odd:

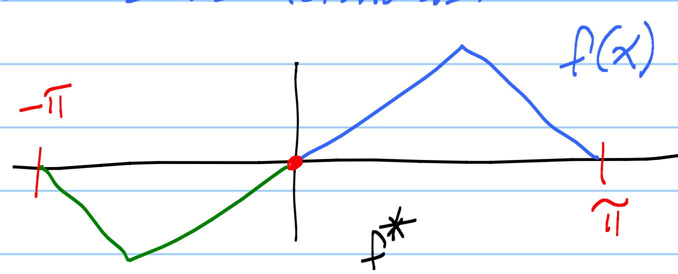
$$\int_{-A}^{A} f(x)\,dx = 0$$

Simple facts:

(even)(even) = even
(even)(odd) = odd
(odd)(odd) = even

Fourier sine series and Fourier cosine series fall out of full Fourier series formulas.

How:



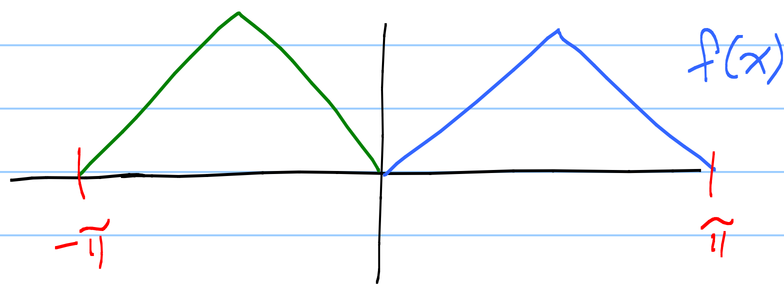
extend to $[-\pi, 0]$ as an odd fcn

Full Fourier series for f^*: All Cosine terms $= 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f^*(x)}_{odd}\ \underbrace{\sin nx}_{odd}}_{even}\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

Aha! Fourier sine series.

Similarly



This time, f^* is <u>even</u> extension to $[-\pi, 0]$.

Full Fourier series for f^*: All sine terms $= 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f^*(x)}_{even}\ \underbrace{\cos nx}_{even}}_{even}\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

Aha! Fourier cosine series on $[0, \pi]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty}{}' \, a_n \cos nx$$

where $\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

Cosine series easier than sine series because no jumps at multiples of π :