Lesson 7 Bessel's inequality

HWK 2 posted now
due Wed, Feb 1, 11:00 pm
in GS

Real inner product: $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$ (f, g real)

Complex inner product: $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx$ (f, g complex)

Bessel's inequality Assume φ_n are continuous real valued fcns on $[-\pi, \pi]$ that are orthonormal, meanig

$$\langle \varphi_n, \varphi_m \rangle = 0 \text{ if } n \neq m \text{ and}$$

$$\langle \varphi_n, \varphi_n \rangle = 1$$

Def^n: The L^2-norm of φ is $\sqrt{\langle \varphi, \varphi\rangle}$

$$= \left(\int_{-\pi}^{\pi} \varphi(x)^2\, dx \right)^{1/2}$$

Hope: $f = \sum_{n=1}^{\infty} c_n \varphi_n$

$$\langle f, \varphi_m \rangle = \left\langle \sum_1^{\infty} c_n \varphi_n, \varphi_m \right\rangle$$

$$= \sum c_n \underbrace{\langle \varphi_n, \varphi_m \rangle}_{\substack{=0 \quad n \neq m \\ =1 \quad n = m}}$$

$$= c_m \cdot 1$$

Very simple formula $\boxed{c_m = \langle f, \varphi_m \rangle}$

Def^n of c_m today.

<u>Big fact</u>: $\sum_{n=1}^{N} c_n \varphi_n$ is the <u>best</u> L^2 approximation to f on $[-\pi, \pi]$ among fcns of the form $\sum_{n=1}^{N} A_n \varphi_n$,

meaning $\|f - \sum_1^N c_n \varphi_n\| \leq \|f - \sum_1^N A_n \varphi_n\|$

with equality <u>only</u> when $c_n = A_n$ for $n = 1, \dots, N$.

Furthermore, $\sum_{n=1}^{N} c_n^2 \leq \|f\|^2$

Aha! $\sum_{n=1}^{\infty} c_n^2$ exists and is $\leq \|f\|^2$.

<u>Thm</u>: $c_n \to 0$ as $n \to \infty$.

<u>Cor</u>: Fourier coeff of a continuous fcn f on $[-\pi, \pi]$ go to zero as $n \to \infty$. $\begin{cases} a_n \to 0 \\ b_n \to 0 \end{cases}$

(So do the complex version of coeff because they are = to simple combos of a_n, b_n.)

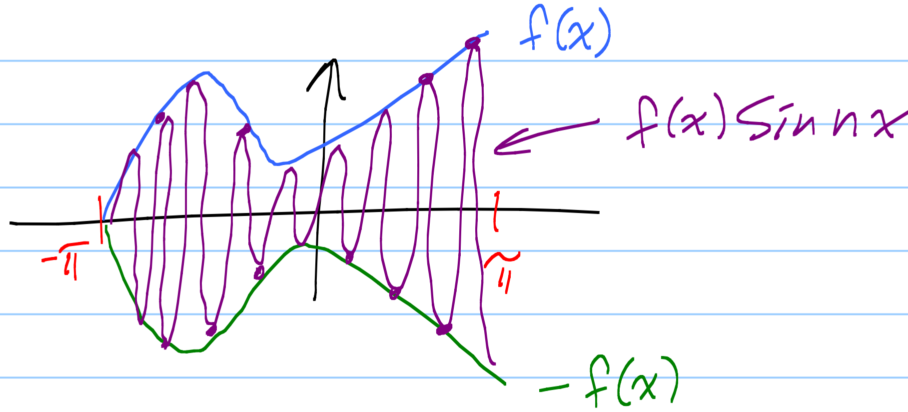
Interesting! Not obvious because

$|a_n| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \right|$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|f(x)|}_{\leq M}\ \underbrace{|\sin nx|}_{\leq 1}\, dx$$

$$\leq \frac{1}{2\pi}(M \cdot 1)\cdot 2\pi = M \qquad \text{ouch!}$$

Gut feeling:



See cancelations of plus hump areas with minus hump areas.

Pf:

$$\overset{E\ =}{E^*} = \Big\| f - \sum_{1}^{N} \overset{A_n}{c_n}\varphi_n \Big\|^2$$

$$= \int_{-\pi}^{\pi} \Big(f - \sum_{n=1}^{N} \overset{A_n}{c_n}\varphi_n\Big)\Big(f - \sum_{m=1}^{N} \overset{A_m}{c_m}\varphi_m\Big)\, dx$$

$$= \int_{-\pi}^{\pi} f^2\, dx - \sum_{n=1}^{N} \overset{A_n}{c_n} \underbrace{\int_{-\pi}^{\pi} f\,\varphi_n\, dx}_{=c_n} - \sum_{m=1}^{N} \overset{A_m}{c_m} \underbrace{\int_{-\pi}^{\pi} f\,\varphi_m\, dx}_{=c_m}$$

$$+ \sum_{n=1}^{N}\sum_{m=1}^{N} \overset{A_n}{c_n}\overset{A_m}{c_m} \underbrace{\int_{-\pi}^{\pi} \varphi_n\varphi_m\, dx}_{=\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}}$$

$$= \int_{-\pi}^{\pi} f^2\, dx - \underbrace{\Big(\sum_{n=1}^{N} c_n^2\Big)}_{S} - S + S$$

Done! $E^* = \| f - \sum_{1}^{N} c_n \varphi_n \|^2 = \int_{-\pi}^{\pi} f^2\,dx - \sum_{n=1}^{N} c_n^2 \geq 0$

$\underbrace{\| f - \sum_{1}^{N} c_n \varphi_n \|^2}_{\geq 0}$

Aha! $\sum_{n=1}^{N} c_n^2 \leq \|f\|^2$ ✓

$$\begin{cases} E = \|f\|^2 - 2\sum_{n=1}^{N} A_n c_n + \sum_{n=1}^{N} A_n^2 \\ E^* = \|f\|^2 - \sum_{n=1}^{N} c_n^2 \end{cases}$$

Aha! $E - E^* = \sum_{n=1}^{N} \Big(\underbrace{c_n^2 - 2 A_n c_n + A_n^2}_{(c_n - A_n)^2} \Big)$

Sum of squares is ≥ 0 !

So $E^* \leq E$. ✓

And, if $E^* = E$, then $(c_n - A_n)^2 = 0$ for $n = 1, 2, \ldots, N$,

So $c_n = A_n$ for $n = 1, 2, \ldots, N$. ✓

Fact: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}$ are orthonormal

Word $\{\varphi_n\}$ is a complete orthonormal basis for $L^2[-\pi,\pi]$ if Bessel's inequality is an equality. (Parseval's identity)

Parseval's : $\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)}$

Hilbert space : $L^2[-\pi,\pi]$ = square integrable fcns on $[-\pi,\pi]$

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\,g(x)\,dx \qquad \|f\| = \sqrt{\int_{-\pi}^{\pi} f^2\,dx}$$

Think: Complete orthonormal basis yields

$$f = \sum_{n=1}^{\infty} c_n \varphi_n \qquad c_n = \langle f,\varphi_n\rangle$$

coordinates!

$$\|f\| \sim \{c_n\}_{n=1}^{\infty} \qquad \sum c_n^2 = \|f\|^2$$

$L^2[-\pi,\pi] \sim \ell^2$ ← square sumable sequences

Parseval's ⟹ this is an isomorphism

$$\|f\|^2 = \|(c_n)_{n=1}^{\infty}\|^2 = \sum c_n^2.$$

ℓ^2 inner product: $\langle (a_n),(b_n)\rangle = \sum_{n=1}^{\infty} a_n b_n$